Practice 02

problem 1 *Equivalence rules*

Consider the following equivalence rules:

- the associativity and commutativity of $\wedge$, $\vee$, and $\oplus$
- double negation: $\boxed{\neg\neg P \equiv P}$
- simplification: $\boxed{P \wedge \bot \equiv \bot}$, $\boxed{P \wedge \top \equiv P}$, $\boxed{P \vee \bot \equiv P}$, and $\boxed{P \vee \top \equiv \top}$
- distribution: $\boxed{A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)}$ and $\boxed{A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)}$
- De Morgan: $\boxed{\neg(A \wedge B) \equiv (\neg A) \vee (\neg B)}$ and $\boxed{\neg(A \vee B) \equiv (\neg A) \wedge (\neg B)}$
- definitions: $\boxed{A \rightarrow B \equiv (\neg A) \vee B}$, $\boxed{(A \leftrightarrow B) \equiv (A \rightarrow B) \wedge (B \rightarrow A)}$ and $\boxed{(A \oplus B) \equiv (A \vee B) \wedge \neg(A \wedge B)}$

Prove that $\boxed{P \rightarrow (A \vee Q) \equiv (P \wedge \neg A) \rightarrow Q}$ by writing out a series of steps, one per line, where the first line is $\boxed{P \rightarrow (A \vee Q)}$, the last line is $\boxed{(P \wedge \neg A) \rightarrow Q}$, and each line other than the first is an application of **one** of those equivalences to the line above it.

You do not need to name your steps, though doing so might help us grade more easily

$P \rightarrow (A \vee Q)$
$(\neg P) \vee (A \vee Q)$
$(\neg P \vee A) \vee Q$
$\neg\neg(\neg P \vee A) \vee Q$
$\neg(\neg P \vee A) \rightarrow Q$
$(\neg\neg P \wedge \neg A) \rightarrow Q$
$(P \wedge \neg A) \rightarrow Q$

Want more practice? Try doing the same for $\boxed{A \oplus B \equiv \overline{A \leftrightarrow B}}$ *and* $\boxed{A \leftrightarrow B \equiv \overline{A \oplus B}}$ *and, if you feel ambitious,* $\boxed{A \oplus B \oplus C \equiv (A \wedge \overline{B} \wedge \overline{C}) \vee (\overline{A} \wedge B \wedge \overline{C}) \vee (\overline{A} \wedge \overline{B} \wedge C) \vee (A \wedge B \wedge C)}$

Expression	Rule used	Why this step?
$A \oplus B$		
$(A \vee B) \wedge \overline{(A \wedge B)}$	definition of $\oplus$	the only option
$((A \wedge \overline{(A \wedge B)}) \vee (B \wedge \overline{(A \wedge B)}))$	distribute	one of two options (De Morgan the other)
$((A \wedge (\overline{A} \vee \overline{B})) \vee (B \wedge (\overline{A} \vee \overline{B})))$	De Morgan	the only option
$((A \wedge \overline{A}) \vee (A \wedge \overline{B})) \vee ((B \wedge \overline{A}) \vee (B \wedge \overline{B}))$	distribute	try to get $A \wedge \overline{A}$ for simplification
$(\bot \vee (A \wedge \overline{B})) \vee ((B \wedge \overline{A}) \vee \bot)$	simplify	
$(A \wedge \overline{B}) \vee (B \wedge \overline{A})$	simplify	
$(\overline{\overline{A}} \wedge \overline{B}) \vee (\overline{\overline{B}} \wedge \overline{A})$	double negation	prep for De Morgan
$\overline{(\overline{A} \vee B)} \vee \overline{(\overline{B} \vee A)}$	De Morgan	need $\vee$ to get $\rightarrow$
$\overline{(\overline{A} \vee B) \wedge (\overline{B} \vee A)}$	De Morgan	need $\wedge$ in middle, not $\vee$
$\overline{(A \rightarrow B) \wedge (B \rightarrow A)}$	definition of $\rightarrow$	
$\overline{A \leftrightarrow B}$	definition of $\leftrightarrow$	

$A \leftrightarrow B$	
$\overline{\overline{A \leftrightarrow B}}$	double negation
…	all steps from above, but backward under a negation
$\overline{A \oplus B}$	

PROBLEM 2 *Prose proof by case analysis*

Write a prose proof of $\boxed{P \oplus Q \equiv \overline{P} \oplus \overline{Q}}$ by completing the provided proof-by-cases template.

Proof. Either P is true or it is false.

Case 1: P **is true** The expression $P \oplus Q$ in this case

is $\top \oplus Q$, which is defined to mean $(\top \vee Q) \wedge \overline{(\top \wedge Q)}$.
Simplifying, that is equivalent to $\top \wedge \overline{Q}$, or simply $\overline{Q}$.

note: this is probably as brief as you should consider being

The expression $\overline{P} \oplus \overline{Q}$ in this case

is $\bot \oplus \overline{Q}$, which is defined to mean $(\bot \vee \overline{Q}) \wedge \overline{(\bot \wedge \overline{Q})}$.
We can simplify that to $(\overline{Q}) \wedge \overline{(\bot)}$,
which is equivalent to $\overline{Q} \wedge \top$ or simply $\overline{Q}$.

note: this is probably as verbose as you should consider being

Because the two are equivalent to the same thing, they are equivalent to each other.

Case 2: P **is false** The expression $P \oplus Q$ in this case

is $\bot \oplus Q$, which is defined to mean $(\bot \vee Q) \wedge \overline{(\bot \wedge Q)}$.
Simplifying, that is equivalent to $Q \wedge \top$, or simply Q.

The expression $\overline{P} \oplus \overline{Q}$ in this case

is $\top \oplus \overline{Q}$, which is defined to mean $(\top \vee \overline{Q}) \wedge \overline{(\top \wedge \overline{Q})}$.
Simplifying, that is equivalent to $\top \wedge \overline{\overline{Q}}$, which is equivalent to Q.

Because the two are equivalent to the same thing, they are equivalent to each other.

Because $P \oplus Q \equiv \overline{P} \oplus \overline{Q}$ is true in both cases, it is true in general. □