Name: ____ CompID: ____________________

Theorem 1 *The sum of all positive odd integers less than $2n$ is n^2.*

PROBLEM 1 *Prove theorem 1 using* **induction**

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Theorem 1 *The sum of all positive odd integers less than $2n$ is n^2.*

PROBLEM 1 *Prove theorem 1 using* ***contradiction*** *and the well-ordering principle*

Name: ______________________________ CompID: ______________

PROBLEM 1 *Fill in these* ***combinatorics*** *blanks*

You may answer any question with factorial, choose, and unresolved arithmetic notation, but may not use ellipses. For example, the following are all OK: $\boxed{120}$, $\boxed{5!}$, $\boxed{\dfrac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1)(3 \cdot 2 \cdot 1)}}$, $\boxed{\dbinom{5}{3}}$.

1. ______________ A seven-character computing ID is 3 letters, 1 digit, and 3 more letters. All 26 letters are used, but digits are limited to 2 through 9 (no 0 or 1). How many seven-character computing ID can this scheme create?

2. ______________ How many 6-element subsets of a 10-element set are there?

3. ______________ Which is larger: $\binom{45}{10}$ or $\binom{45}{50}$?

4. ______________ How many 6-element sequences can be made from elements of a 50-element set without repeating elements?

5. ______________ How many 6-element sequences can be made from elements of a 50-element set where no element can appear twice in a row? For example, $(1, 2, 1, 2, 1, 2)$ is OK, but $(1, 2, 2, 1, 2, 1)$ is not OK.

6. ______________ If I randomly shuffle a list containing 10 `d`s and 16 `x`s, what is the probability the shuffle will result in the exact sequence `"ddddddddddxxxxxxxxxxxxxxxx"`?

7. ______________ In a fair raffle, every participant has an equal chance of winning. I participate in two fair raffles: one with 10 people (myself included), one with 100 (myself included). What is my chance of winning at least one raffle?

8. ______________ Which adds more options when constructing sequences: doubling the number of options for each spot in the sequence or doubling the length of the sequence? Answer with one of **options**, **length**, or **same**. You may assume both the options and length are initially at least 2.

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Consider the following **sets**: $A = \{8, 4, 5\}$, $B = \{2, 3, 4\}$, $C = \mathcal{P}\big(\{8, 2\}\big)$

PROBLEM 1 *Show all members of each set*

1. ____ $= C$

2. ____ $= A \cup B$

3. ____ $= A \cap B$

4. ____ $= A \setminus B$

5. ____ $= \big\{3x \mid (x \in \mathbb{N}) \wedge (2x \in A)\big\}$

6. ____ $= \{1\} \cap \mathcal{P}\big(\{1\}\big)$

7. ____ $= \big\{x \mid (x \in A) \wedge (2x \in B)\big\}$

8. ____ $= \Big\{\{a, b\} \;\Big|\; (a \in A) \wedge \big(b \in \{4, 5\}\big)\Big\}$

PROBLEM 2 *Answer each question*

9. ______________ $= |A|$

10. ______________ $= \big|\mathcal{P}(A)\big|$

11. ______________ $= \Big|\mathcal{P}\big(\mathcal{P}(A)\big)\Big|$

12. ______________ $= 8 \in A$

13. ______________ $= \{8\} \in A$

14. ______________ $= \big\{\{8\}\big\} \in A$

Name: ______________________________ CompID: ______________

Consider the following **discrete structures** questions.

PROBLEM 1 *Write out in full*

1. ______________________________ $= \{1\} \times \{1\} \times \{2, 3\}$

2. ______________________________ $= \left|\{1, 2\}^4\right|$

3. ______________________________ = all the subsequences of `"ook"`

4. ______________________________ = a subsequence of `"fun"` that is not a substring of `"fun"`

5. ______________________________ = the image of $\{0, 3\}$ under $R(x) = x + 2$

6. ______________________________ = the set of edges of the graph ①→②⇄③

PROBLEM 2 *Draw*

7. ◯⟶◯⇄◯⟵◯ add a minimal number of edges to make this the graph of a transitive relation

8. ◯⟶◯⇄◯⟵◯ add a minimal number of edges to make this the graph of a reflexive relation

PROBLEM 3 *Logarithms*

9. Simplify $\log_2(5) + \log_2(3)$: ______________________________

10. Re-write $\log_3(x^q)$ without exponentiation: ______________________________

11. Re-write $\log_4(x)$ using base-3 log(s) instead of base-4: ______________________________

12. Fill in the blank: $\log_4(9) = \log_2\left(\underline{\qquad\qquad}\right)$

Name: ____ CompID: ____________________

Consider the following **logic** questions. You do not need to specify your domains, propositions, or predicate definitions, though you may if you wish.

PROBLEM 1 *Convert the underlined parts to logic*

1. "we know that any quiznidic number is prime"

2. "are there any three-element sets in Q?"

3. "Every woozle is a dingalo."

4. "At least one hefalump is a bear."

PROBLEM 2 *Convert to English*

domain: all animals
$M(x)$: $\_\_\_\__x$ is a monkey
$L(x, y)$: $\_\_\_\__x$ loves $\_\_\_\__y$
p: Peevy

5. Write a clear English sentence that means $\forall x \,.\, \exists y \neq x \,.\, L(x, y)$.

6. Write a clear English sentence that means $\forall x, y \,.\, L(x, p) \vee \big(M(y) \rightarrow L(y, x)\big)$.

(*continued on reverse*)

PROBLEM 3 *Apply axioms*

Show that $\big((P \wedge Q) \vee (K \wedge M)\big) \vdash (P \vee K)$ by direct proof and/or proof by cases. You may mix math and English if you wish; we are looking for sound logic, not prose proof technique.

Symbols

Concept	Java/C	Python	This class	Bitwise	Other
true	`true`	`True`	$\top$ or 1	`-1`	T, tautology
false	`false`	`False`	$\bot$ or 0	`0`	F, contradiction
not P	`!p`	`not p`	$\neg P$ or $\overline{P}$	`~p`	
P and Q	`p && q`	`p and q`	$P \wedge Q$	`p & q`	PQ, $P \cdot Q$
P or Q	`p || q`	`p or q`	$P \vee Q$	`p | q`	$P + Q$
P xor Q	`p != q`	`p != q`	$P \oplus Q$	`p ^ q`	$P \veebar Q$
P implies Q			$P \rightarrow Q$		$P \supset Q$, $P \Rightarrow Q$
P iff Q	`p == q`	`p == q`	$P \leftrightarrow Q$		$P \Leftrightarrow Q$, P xnor Q, $P \equiv Q$

Concept	Symbol	Meaning
equivalent	$\equiv$	"$A \equiv B$" means "$A \leftrightarrow B$ is a tautology"
entails	$\vDash$	"$A \vDash B$" means "$A \rightarrow B$ is a tautology"
provable	$\vdash$	"$A \vdash B$" means both "$A \vDash B$" and "I know B is true because A is true""$\vdash B$" (i.e., without A) means "I know B is true"
therefore	$\therefore$	"$\therefore A$" means both "$\vdash A$" and "A is the thing we wanted to show"

Graphs and Relations

Term	Definition
Walk	An alternating sequence of vertices and edges • starting and ending with a vertex, • each edge (x, y) in the walk follows vertex x and is followed by vertex y
Path	A walk that does not visit any vertex twice
Closed Walk	A walk that begins and ends at the same vertex
Cycle	A closed walk that is a path except for its last vertex

The related definitions on relations $R : A \rightarrow A$ are

Term	Definition
R is Reflexive	$\forall x \in A \,.\, x \text{ R } x$
R is Irreflexive	$\forall x \in A \,.\, \neg(x \text{ R } x)$
R is Symmetric	$\forall x, y \in A \,.\, (x \text{ R } y) \rightarrow (y \text{ R } x)$
R is Asymmetric	$\forall x, y \in A \,.\, (x \text{ R } y) \rightarrow \neg(y \text{ R } x)$
R is Antisymmetric	$\forall x \neq y \in A \,.\, (x \text{ R } y) \rightarrow \neg(y \text{ R } x)$
R is Transitive	$\forall x, y, z \in A \,.\, (x \text{ R } y) \wedge (y \text{ R } z) \rightarrow (x \text{ R } z)$

And those lead to these terms:

Term	Definition
Strict partial order	transitive and asymmetric
Weak partial order	transitive, reflexive, and antisymmetric
Equivalence relation	transitive, reflexive, and symmetric

The following operators are both **associative** (you can add and remove parentheses around them) and **commutative** (you can swap their operands' position): $\wedge$, $\vee$, $\oplus$

The following operator is *commutative* but not *associative*: $\leftrightarrow$

form 1	form 2	Name of rule
$A \rightarrow B$	$\neg A \vee B$	
$A \wedge (B \vee C)$	$(A \wedge B) \vee (A \wedge C)$	Distributive law
$A \vee (B \wedge C)$	$(A \vee B) \wedge (A \vee C)$	Distributive law
$\neg(A \wedge B)$	$(\neg A) \vee (\neg B)$	De Morgan's law
$\neg(A \vee B)$	$(\neg A) \wedge (\neg B)$	De Morgan's law
$(A \leftrightarrow B)$	$(A \rightarrow B) \wedge (B \rightarrow A)$	
$(A \oplus B)$	$(A \vee B) \wedge \neg(A \wedge B)$	

form 1	form 2	Name of rule
$A \oplus B$	$\neg(A \leftrightarrow B)$	
$A \leftrightarrow B$	$\neg(A \oplus B)$	xnor
$P \rightarrow (A \vee Q)$	$(P \wedge \neg A) \rightarrow Q$	

Given	Entails	Names
$\forall x \in S \;.\; P(x)$	$P(s)$, for any $s \in S$ we care to pick	universal instantiation
$\exists x \in S \;.\; P(x)$	$s \in S \wedge P(s)$ where s is an otherwise-undefined new variable	existential instantiation
$s \in S \vdash P(s)$	$\forall x \in S \;.\; P(x)$	universal generalization
$P(s) \wedge s \in S$	$\exists x \in S \;.\; P(x)$	existential generalization

Given	Entails	Name
	$A \vee \neg A$	excluded middle
$A \wedge B$	A	
A and B	$A \wedge B$	
A	$A \vee B$	
$A \vee B$ and $\neg B$	A	disjuctive syllogism
$A \rightarrow B$ and $B \rightarrow C$	$A \rightarrow C$	hypothetical syllogism; transitivity of implication
$A \rightarrow B$ and A	B	modus ponens
$A \rightarrow B$ and $\neg B$	$\neg A$	modus tolens
$A \leftrightarrow B$	$A \rightarrow B$	
$A \rightarrow C$, $B \rightarrow B$, and $A \vee B$	C	
$A \rightarrow B$, $C \rightarrow D$, and $A \vee C$	$B \vee D$	
$A \rightarrow B$	$A \rightarrow (A \wedge B)$	
$\neg(A \wedge B)$, A	$\neg B$	

A proof that assumes A and derives B entails that $A \rightarrow B$.
A proof that assumes A and derives $\bot$ entails that $\neg A$.
$\log_{a^b}(x) = b^{-1} \log_a(x)$
$(a \in \mathbb{Z}) \wedge (a > 1) \vDash (a$ has at least two factors$)$
$(a \in \mathbb{Z}) \wedge (a > 1) \wedge (a$ has exactly two factors$) \equiv (a$ is prime$)$