Name: ____________________ CompID: ____________________

CS 2102 - DMT1 - Spring 2020 — Luther Tychonievich
Administered in class friday march 27, 2020

Quiz 08

problem 1 *Summation proofs*

Prove the following theorems by induction.

1. $\sum_{i=1}^{n} \frac{2}{3^n} = \frac{3^n - 1}{3^n}$

Proof.

This theorem is false except when $n = 1$. Consider the case where $n = 2$:

$$\sum_{i=1}^{2} \frac{2}{3^n} = \frac{2}{3^2} + \frac{2}{3^2} = \frac{4}{9} \neq \frac{3^2 - 1}{3^2} = \frac{8}{9}$$

The correct convergence of $\sum_{i=1}^{n} \frac{2}{3^n}$ is $\sum_{i=1}^{n} \frac{2}{3^n} = \frac{1}{3^n} \sum_{i=1}^{n} 2 = \frac{2n}{3^n}$ which can be shown directly via algebra; no induction needed.

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The slightly different theorem with 3^i in the denominator is true: $\sum_{i=1}^{n} \frac{2}{3^i} = \frac{3^n - 1}{3^n}$

Proof.
We proceed by induction.

Base Case When $n = 1$ we have $\sum_{i=1}^{2} \frac{2}{3^i} = \frac{2}{3} = \frac{3^1 - 1}{3^i}$.

Inductive step Assume the theorem holds for some $k \in \mathbb{Z}^+$: that is, $\sum_{i=1}^{k} \frac{2}{3^i} = \frac{3^k - 1}{3^k}$. Adding $\frac{2}{3^{k+1}}$ to both sides we get $\frac{2}{3^{k+1}} + \sum_{i=1}^{k} \frac{2}{3^i} = \frac{3^k - 1}{3^k} + \frac{2}{3^{k+1}}$; the left-hand side is equivalent to $\sum_{i=1}^{k} \frac{2}{3^i}$ and the right-hand side can be re-written as

$$\begin{aligned} & \frac{3^k - 1}{3^k} + \frac{2}{3^{k+1}} \\ = & \frac{(3^k - 1) \cdot 3}{3^k \cdot 3} + \frac{2}{3^k \cdot 3} \\ = & \frac{(3^{k+1} - 3) + 2}{3^{k+1}} \\ = & \frac{3^{k+1} - 1}{3^{k+1}} \end{aligned}$$

which is the theorem at $n = k + 1$.

By the principle of induction, the revised theorem holds for all $n \in \mathbb{N}$. □