CS 2102 - DMT1 - Spring 2020 — Luther Tychonievich
Practice exercise in class friday march 6, 2020

Practice 07

problem 1 *Convert to prose*

S: the set of all snakes
R: the set of all rabbits
$E(x,y)$: x eats y
$Y(x)$: x is yellow

Convert the following to simple, readable English:

1. $(\exists r \in R, s \in S \,.\, E(r,s)) \rightarrow (\neg\forall s \in S \,.\, \exists r \in R \,.\, E(s,r))$

If any rabbit eats a snake then not all snakes have a rabbit they eat.

2. $\forall r \in R, s \in S \,.\, (Y(s) \rightarrow \neg E(s,r)) \wedge (Y(r) \rightarrow E(r,s))$

Yellow snakes don't eat rabbits, but yellow rabbits eat all the snakes.

3. $\forall s_1 \in S \,.\, \exists s_2 \in S \,.\, \forall s_3 \in S \,.\, Y(s_1) \rightarrow (\neg E(s_2,s_3) \wedge E(s_1,s_2) \wedge \neg Y(s_2))$

Every yellow snakes eats some non-yellow snake that doesn't eat snakes.

problem 2 *Primes and factors*

4. $2^2 \cdot 7$ is the prime factorization of 28

5. 2^8 is the prime factorization of 256

6. 31 is the prime factorization of 31

7. $2^{25} \cdot 7^9$ is the prime factorization of $4^8 \cdot 14^9$

8. $\{1,2,4,7,8\}$ is the set positive 1-digit numbers relatively prime with 15

9. $\{1,2,4,5,7,8\}$ is the set positive 1-digit numbers relatively prime with 81

PROBLEM 3 *Symbolic proof by contradiction*

Write a symbolic proof outline of the the following, using proof-by-contradiction.

10. $\frac{2}{3} \notin \mathbb{Z}$

Assume $\frac{2}{3} \in \mathbb{Z}$

$\exists x \in \mathbb{Z} \,.\, \frac{2}{3} = x$	definition of set membership
$\frac{2}{3} = x$	existential instantiation
$2 = 3x$	algebra
3 is a factor of 2	fundamental theorem of arithmetic
$\perp$	contradiction
Ergo assumption false	proof by contradiction
$\frac{2}{3} \notin \mathbb{Z}$	conclusion

11. $\sqrt{2} \notin \mathbb{Q}$

Assume $\sqrt{2} \in \mathbb{Q}$

$\exists x, y \in \mathbb{Z} \,.\, \sqrt{2} = \frac{x}{y}$	definition of rationals
$\sqrt{2} = \frac{x}{y}$	existential instantiation
$2y^2 = x^2$	algebra
2 is a factor of $2y^2$ with odd multiplicity	fundamental theorem of arithmetic
all factors of x^2 have even multiplicity	fundamental theorem of arithmetic
$\perp$	contradiction
Ergo assumption false	proof by contradiction
$\sqrt{2} \notin \mathbb{Q}$	conclusion

PROBLEM 4 *Prose from symbols*

Write a prose proof that follows the given symbolic proof outlines.

Assume $\frac{5}{8} \in \mathbb{Z}$	
$\exists x \in \mathbb{Z} \,.\, \frac{5}{8} = x$	definition of set membership
$\frac{5}{8} = x$	existential instantiation
$5 = 8x$	algebra
12. 2 is a factor of 5	fundamental theorem of arithmetic
$\bot$	contradiction
Ergo assumption false	proof by contradiction
$\frac{5}{8} \notin \mathbb{Z}$	conclusion

Proof.

We proceed by contradiction.
Assume that $\frac{5}{8}$ is an integer; call that integer x. This means that $5 = 8x$. By the fundamental theorem of algebra, both sides must have the same prime factors; in particular, 2 is a factor of $8x$, so it must be a factor of 5; but 2 is not a factor of 5, giving us a contradiction.
Because assume that $\frac{5}{8}$ is an integer resulted in a contradiction, it must be the case that $\frac{5}{8}$ is not an integer.
□

	Assume $\sqrt[3]{4} \in \mathbb{Q}$	
	$\exists x, y \in \mathbb{Z} \,.\, \sqrt[3]{4} = \frac{x}{y} \wedge \gcd(x, y) = 1$	definition of set rationals
	$\sqrt[3]{4} = \frac{x}{y}$	existential instantiation
	$4y^3 = x^3$	algebra
	$\neg(2 \mid x) \vee \neg(2 \mid y)$	because $\gcd(x, y) = 1$
	case 1: $\neg(2 \mid x)$	
	$\neg(2 \mid x^3)$	
	$\bot$	
13.	case 2: $\neg(2 \mid y)$	
	$(2 \mid x^3)$	case analysis
	$(2 \mid x)$	
	$(8 \mid x^3)$	
	$\neg(8 \mid 2y^3)$	
	$\bot$	
	$\bot$	contradiction
	Ergo assumption false	proof by contradiction
	$\sqrt[3]{4} \notin \mathbb{Q}$	conclusion

Proof.

We proceed by contradiction.
Assume that $\sqrt[3]{4}$ is a rational number; write that rational in lowest terms as $\frac{x}{y}$. This means that $4y^3 = x^3$. Because $\frac{x}{y}$ is in lowest terms, 2 cannot be a factor of both x and y; we thus consider two cases:

Case: 2 is not a factor of x This contradicts the fundamental theorem of arithmetic: because $4y^3 = x^3$, 2 must be a factor of x^3 and hence a factor of x as well.

Case: 2 is not a factor of y By the fundamental theorem of arithmetic, 2 must be a factor of x^3 and hence 8 must be a factor of x^3; however, 8 cannot be a factor of $4y^3$ unless 2 is a factor of y, resulting in a contradiction.

Because both cases resulted in a contradiction, we have a contradiction in general.
Because assume that $\sqrt[3]{4}$ is a rational number resulted in a contradiction, it must be the case that $\sqrt[3]{4}$ is irrational. □

PROBLEM 5 *Proof by contradiction*

Prove the following using proof-by-contradiction. You may prove them in prose or in symbols or any readable mix of the two.

14. $\sqrt{2} \notin \mathbb{Z}$

Proof.

We proceed by contradiction.
Assume $\sqrt{2} \in \mathbb{Z}$; let $x \in \mathbb{Z}$ be the element of $\mathbb{Z}$ that equals $\sqrt{2}$. Thus, $2 = x^2$, which means that the prime factorization of x is 2^1. But a square must have even powers and 1 is not even, which is a contradiction.
Because assuming $\sqrt{2} \in \mathbb{Z}$ led to a contradiction, it must be the case that $\sqrt{2} \notin \mathbb{Z}$. □

15. $2^{-1} \notin \mathbb{Z}$

Proof.

We proceed by contradiction.
Assume 2^{-1} is an integer; call that integer x. Then $2^{-1} = x$, meaning $1 = 2x$. By the fundamental theorem of arithmetic, that means that 2 is a factor of 1, but it is not.
Because assuming 2^{-1} is an integer led to a contradiction, it must be the case that $2^{-1} \notin \mathbb{Z}$. □

16. $\sqrt{7} \notin \mathbb{Q}$

Proof.

We proceed by contradiction.
Assume $\sqrt{7} \in \mathbb{Q}$. Then

$$\begin{aligned} \exists x, y \in \mathbb{Z} \, . \, \frac{x}{y} &= \sqrt{7} \\ \frac{x}{y} &= \sqrt{7} \\ x &= \sqrt{7}y \\ x^2 &= 7y^2 \end{aligned}$$

But x^2 must have an even number of 7s in its prime factorization and $7y^2$ must have an odd number, which is a contradiction.
Because assuming $\sqrt{7} \in \mathbb{Q}$ led to a contradiction, it must be the case that $\sqrt{7} \notin \mathbb{Q}$. □

17. $3^{1.5} \notin \mathbb{Q}$

Proof.

We proceed by contradiction.
Assume $3^{1.5} \in \mathbb{Q}$. Let $\frac{x}{y} = 3^{1.5}$, where x and y are coprime. Then $\left(\frac{x}{y}\right)^2 = 3^3$ meaning $x^2 = 3^3y^2$. x^2 has an even number of 3s in its prime factorization, as does y^2, meaning 3^3y^2 has an odd number. But prime factorization are unique, meaning equal values cannot have differing numbers of 3s in their factorization: thus we have a contradiction.
Because assuming $3^{1.5} \in \mathbb{Q}$ led to a contradiction, it must be the case that $3^{1.5} \notin \mathbb{Q}$. □

PROBLEM 6 *Additional problems*

18. Prove there are infinitely many prime numbers. Use $p' = 1 + \prod_{p \in P} p$ where P is the set of all primes to derive the contradiction (e.g. by showing both that $p' \in P$ and $p' \notin P$).

19. Prove there are infinitely many integers. Use $z + 1$ where z is the largest integer to derive the contradiction.

20. Prove there are infinitely many finite-length strings containing the digits `0` and `1`. Use the concatenation of s and s, where z a one of the strings of maximal length, to derive the contradiction.

21. Prove there are infinitely many finite natural numbers. Use $n + 1$, where n is the largest finite natural number, to derive the contradiction.

22. Prove that $\forall n \in \mathbb{N} \;.\; 4|(5^n - 1)$. Use the well-ordering principle to derive a contradiction by showing that if $m > 0$ is the smallest n that makes the expression false, then $m - 1$ also makes it false. Include a case that shows that the expression holds for $n = 0$.

23. Prove that $\forall n \in \mathbb{Z}^+ \;.\; \overline{p_1 \wedge p_2 \wedge \ldots \wedge p_n} \equiv \overline{p_1} \vee \overline{p_2} \vee \ldots \vee \overline{p_n}$. Use the well-ordering principle to derive a contradiction by showing that if $m > 1$ is the smallest n that makes the expression false, then $m - 1$ also makes it false. Include a case that shows that the expression holds for $n = 1$.

24. Prove there is no smallest positive real number. Use the well-ordering principle to derive a contradiction by showing a smaller positive real number than the smallest positive real. Tools like $n \div 2$ or $n \times n$ might help.

25. Prove there is no real number that is closest to, but not the same as, x. Use the well-ordering principle to derive a contradiction by showing a closer real number than the closest real. Tools like $\frac{x+y}{2}$ might help.

26. Prove there is no best rational approximation of $\sqrt{2}$ by showing that, for every approximation x, the value $\frac{x}{2} + \frac{1}{x}$ is a better approximation; you may need to a lemma to show that that $\forall x \in \mathbb{Q} \;.\; \frac{x}{2} + \frac{1}{x} \neq x$.

27. Prove that $\forall x \in \mathbb{Z} \;.\; (x + 1)(x - 1) = x^2 - 1$ without using the distributive law of multiplication. Instead show that it holds for some x (pick any you wish) and that there's no largest or smallest x for which it does not hold.

28. Prove that there is no largest two-argument function `f(x, y)` that returns `x + y` in the programming language of your choice. Do this by showing that if there was a largest program, you can make a larger one that has the same behavior.

29. Prove that there is no most-complicated two-argument function `f(x, y)` that returns `x + y` in the programming language of your choice, where complication is measured by the number of `if` statements and loops. Do this by showing that if there was a most complicated program, you can make a more complicated one that has the same behavior.

30. Prove that there is no longest-running two-argument function `f(x, y)` that returns `x + y` in the programming language of your choice. Do this by showing that if there was a most longest-running program, you can make a program that takes longer to execute and has the same behavior.