CS 2102 - DMT1 - Spring 2020 — Luther Tychonievich
Practice exercise in class friday march 6, 2020

Practice 07

problem 1 *Products and Powers*

Write out the following in full.

1. $\{1,2\} \times \{3\} \times \{1,4\} =$ $\{(1,3,1),(1,3,4),(2,3,1),(2,3,4)\}$

2. $\{56\}^3 =$ $\{(56,56,56)\}$

3. $\{1,2\} \times \mathcal{P}(\{1\}) =$ $\big\{(1,\{\}),(1,\{1\}),(2,\{\}),(2,\{1\})\big\}$

4. $\{(1,2)\} \times \mathcal{P}(\{(1,2,3,4)\}) =$ $\{(1,2,\{\}),(1,2,\{(1,2,3,4)\})\}$; some uses will treat singleton sets as their one element to get $\{(1,2,\{\}),(1,2,1,2,3,4)\}$ or chose not to flatten to get $\{((1,2),\{\}),((1,2),\{(1,2,3,4)\})\}$ or $\{((1,2),\{\}),((1,2),(1,2,3,4))\}$ instead.

5. $\{\mathtt{a},\mathtt{b}\}^2 =$ {"aa", "ab", "ba", "bb"}

6. $\{4,1\} \times \{1,2\} =$ $\{(4,1),(4,2),(1,1),(1,2)\}$

7. $\{4\} \times \{1,2\} \times \{3\}^3 =$ $\{(4,1,3,3,3),(4,2,3,3,3)\}$

8. $\mathcal{P}(\{\})^2 =$ $\{(\{\},\{\})\}$

problem 2 *Members of Products and Powers*

Give two different example members of each of the following sets. Make them different from one another: different lengths, different internal patterns, etc., is the set allows that. If there are not enough elements of the set to give two different elements, leave some blanks blank.

9. $\{\mathtt{a},\mathtt{b},\mathtt{c}\}^4$ contains "aaaa" and "cbac"

10. $\{\mathtt{a},\mathtt{b},\mathtt{c}\}^1$ contains "a" and "c"

11. $\{\mathtt{a},\mathtt{b},\mathtt{c}\}^0$ contains "" and ______

12. $\{\mathtt{a},\mathtt{b},\mathtt{c}\}^*$ contains "ba" and "cababacc"

13. {"`good`", "`fun`"}^{2} contains "goodgood" and "fungood"

Give two strings of length 3 belonging to

14. {"`a`", "`ok`"}*: "aaa" and "aok"

15. {"`a`", "`bb`", "`ccc`"}*: "bba" and "ccc"

PROBLEM 3 *Subsequences*

Definition 1 *A **subsequence** is a sequence that can be derived from another sequence by deleting zero or more elements without changing the order of the remaining elements.*

What are the subsequences of the string "`OK`"? "", "O", "K", "OK"

What is the longest subsequence shared by "`MATHEMATICS`" and "`COMPUTERS`"? "MTES"

PROBLEM 4 *Summation proofs*

Prove the following theorems by induction.

16. $\forall n \in \mathbb{N} \ . \ \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$

Proof.

We proceed by induction.

Base Case When $n = 0$ we have $\sum_{i=0}^{0} i = 0$ and $\frac{(0)(1)}{2} = 0$, so the theorem holds for $n = 0$.

Inductive step Assume the theorem holds for some $n \in \mathbb{N}$: that is, $\sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$. Adding $n+1$ to both sides, we have $n + 1 + \sum_{i=0}^{n} i = n + 1 + \frac{(n)(n+1)}{2}$; the left-had side is equivalent to $\sum_{i=0}^{n+1} i$ by the definition of summation; the right-hand side can be rearranged using algebra to get $\frac{2(n+1)+(n)(n+1)}{2} = \frac{(2+n)(n+1)}{2} = \frac{(n+1)((n+1)+1)}{2}$; this means that $\sum_{i=0}^{n+1} i = \frac{(n+1)((n+1)+1)}{2}$, or in other words that the theorem holds for $n+1$.

By the principle of induction, the theorem holds for all $n \in \mathbb{N}$. □

17. $\forall n \in \mathbb{N} \,.\, \sum_{x=0}^{n} \frac{1}{2^x} = \frac{2^{n+1}-1}{2^n}$

Proof.

We proceed by induction.

Base Case When $n = 0$ we have $\sum_{x=0}^{0} \frac{1}{2^x} = 1$ and $\frac{2^1 - 1 = 1}{2^0 = 1} = 1$, so the theorem holds for $n = 0$.

Inductive step Assume the theorem holds for some $n \in \mathbb{N}$: that is, $\sum_{x=0}^{n} \frac{1}{2^x} = \frac{2^{n+1}-1}{2^n}$. Adding $\frac{1}{2^{n+1}}$ to both sides, we have $\frac{1}{2^{n+1}} + \sum_{x=0}^{n} \frac{1}{2^x} = \frac{1}{2^{n+1}} + \frac{2^{n+1}-1}{2^n}$; the left-had side is equivalent to $\sum_{x=0}^{n+1} \frac{1}{2^x}$ by the definition of summation; the right-hand side can be rearranged to get $\frac{1+2(2^{n+1}-1)}{2^{n+1}} = \frac{2^{n+2}-1}{2^{n+1}}$; this means that $\sum_{x=0}^{n+1} \frac{1}{2^x} = \frac{2^{n+2}-1}{2^{n+1}}$, or in other words that the theorem holds for $n + 1$.

By the principle of induction, the theorem holds for all $n \in \mathbb{N}$. □

18. $\forall n \in \mathbb{N} \; . \; \sum_{x=n}^{2n} x = \frac{3(n+1)n}{2}$

Proof.

We proceed by induction.

Base Case When $n = 0$ we have $\sum_{x=0}^{0} 0 = 0$ and $\frac{3(0)9}{2} = 0$, so the theorem holds for $n = 0$.

Inductive step Assume the theorem holds for some $n \in \mathbb{N}$: that is, $\sum_{x=n}^{2n} x = \frac{3(n+1)n}{2}$. Consider the sum evaluated at $n+1$:

$$\begin{aligned}
\sum_{x=n+1}^{2(n+1)} x &= -n + 2n + 1 + 2n + 2 + \sum_{x=n}^{2n} x \\
&= 3n + 3 + \sum_{x=n}^{2n} x \\
&= 3n + 3 + \frac{3(n+1)n}{2} \\
&= 3n + 3 + \frac{3n^2 + 3n}{2} \\
&= \frac{6n + 6 + 3n^2 + 3n}{2} \\
&= \frac{3(n^2 + 3n + 2)}{2} \\
&= \frac{3(n+2)(n+1)}{2} \\
&= \frac{3((n+1)+1)(n+1)}{2}
\end{aligned}$$

which means the theorem holds at $n+1$ as well.

By the principle of induction, the theorem holds for all $n \in \mathbb{N}$. □

19. $\forall x \in \{a \mid a \in \mathbb{Z} \wedge a \geq -1\} . \sum_{k=-1}^{x} 12 - 2k = 26 + 11x - x^2$

Proof.

We proceed by induction.

Base Case When $x = -1$ we have $\sum_{k=-1}^{-1} 12 - 2k = 14 = 26 - 11 - 1$, so the theorem holds for $x = -1$.

Inductive step Assume the theorem holds for some x; that is, $\sum_{k=-1}^{x} 12 - 2k = 26 + 11x - x^2$. Consider the sum evaluated at $x + 1$:

$$\begin{aligned}
\sum_{k=-1}^{x+1} 12 - 2k &= 12 - 2(x+1) + \sum_{k=-1}^{x} 12 - 2k \\
&= 10 - 2x + 26 + 11x - x^2 \\
&= (11 - 1) - 2x + 26 + 11x - x^2 \\
&= 26 + (11 + 11x) - (1 + 2x + x^2) \\
&= 26 + 11(x + 1) - (x + 1)^2
\end{aligned}$$

which means the theorem holds at $x + 1$ as well.

By the principle of induction, the theorem holds for all $x \in \{a \mid a \in \mathbb{Z} \wedge a \geq -1\}$. □

You might also try doing inductive proofs with other summation formulae, such as

$$
\begin{aligned}
\sum_{i=0}^{n} i^2 &= \frac{(n+1)(2n+1)(n)}{6} \\
\sum_{i=1}^{n+1} i^2 &= \frac{(n+2)(2n+3)(n+1)}{6} \\
\sum_{i=2}^{n+2} i^2 &= \frac{(n+3)(2n+5)(n+2)}{6} \\
6\sum_{i=0}^{n} i^3 - i &= \binom{n+2}{4} \\
\sum_{x=0}^{n} \frac{x^2-1}{x+1} &= \frac{(n+1)(n-1)}{2} \\
\sum_{x=0}^{n} x^3 - x^2 &= \frac{(n+1)(3n+2)(n)(n-1)}{12} \\
\sum_{i=0}^{n} 3i^2 + 2i &= \frac{(2n+3)(n+1)(n)}{2} \\
\sum_{x=n}^{n^2} x &= \frac{n+n^4}{2} \\
\sum_{x=0}^{2n} (-1)^x x &= n \\
\sum_{i=1}^{n} \frac{1}{2^i} &= \frac{2^n-1}{2^n} \\
\sum_{k=-n}^{0} k &= \frac{(n+1)n}{-2} \\
\sum_{i=1}^{n} \frac{1}{3^i} &= \frac{3^n-1}{3^n 2} \\
\forall k \neq 1 \,.\, \left(\sum_{i=1}^{n} \frac{1}{k^i} \right. &= \left. \frac{k^n-1}{k^n(k-1)} \right)
\end{aligned}
$$

Note: at least one of the above formulae is false. In the process of proving it you should find the normal methods not working, revealing the non-truth.