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Pledge: _____

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Signature: _____

There are 180 minutes (3 hours) for this exam and 180 points on the test; don't spend too long on any one question!

You may use scrap paper. However, all answers must be on these ten exam pages.

Part I	_____/40
Part II	_____/45
Part III	_____/20
Part IV	_____/30
Part V	_____/35
Part VI	_____/10
Total	_____/180

Part I: Mathematical reasoning

1. (40 points total) Consider the following four statements. Note that, for this example, the negation of "richly colored" is "dully colored", and "small" means the same as "not large".

> All hummingbirds are richly colored
> No large birds live on honey
> Birds that do not live on honey are dull in color
> Hummingbirds are small

a. (10 points) Express each of the four statements using quantifiers and propositional functions. Clearly label what your propositional functions represent, as well as your universe of discourse. Your universe of discourse must be the same for all the quantified statements.

Answer:

$\forall x(P(x) \rightarrow S(x))$ $P(x)$ means x is a hummingbird
$\neg \exists x(Q(x) \wedge R(x))$ $Q(x)$ means x is large
$\forall x(\neg R(x) \rightarrow \neg S(x))$ $R(x)$ means x lives on honey
$\forall x(P(x) \rightarrow \neg Q(x))$ $S(x)$ means x is richly colored
                         The universe of discourse is all birds

b. (5 points) If they are not already, convert the quantified statements to use only one universal quantifier per statement (for example, $\forall x(P(x) \rightarrow Q(x))$). If you don't remember how to convert existential quantified statements into universally quantified statements, then try rephrasing the above sentences using only universal quantifiers.

Answer:

$\forall x(P(x) \rightarrow S(x))$ $P(x)$ means x is a hummingbird
$\forall x(\neg Q(x) \vee \neg R(x))$ $Q(x)$ means x is large
$\forall x(\neg R(x) \rightarrow \neg S(x))$ $R(x)$ means x lives on honey
$\forall x(P(x) \rightarrow \neg Q(x))$ $S(x)$ means x is richly colored
                         The universe of discourse is all birds

c. (5 points) As all the statements in (b) now use one universal quantifier, we convert them to use only propositions. For example, $\forall x(P(x) \rightarrow Q(x))$ would become $p \rightarrow q$. Express these statements using only propositions. If you did not get part (b), try restating the sentences using propositions.

Answer:

$p \rightarrow s$ $\neg r \rightarrow \neg s$
$\neg q \vee \neg r$ $p \rightarrow \neg q$

d. (20 points) If you are given that the first three statements from part (c) are true, can you conclude the fourth statement? Prove this using logical equivalences and rules of inference. Clearly label what rule you are using on each step.

Answer:

1	$p \rightarrow s$	1st hypothesis
2	$\neg r \rightarrow \neg s$	3rd hypothesis
3	$s \rightarrow r$	Contrapositive of step 2
4	$p \rightarrow r$	Hypothetical syllogism of steps 1 and 3
5	$\neg q \vee \neg r$	2nd hypothesis
6	$\neg r \vee \neg q$	Commutative law from step 5
7	$r \rightarrow \neg q$	Definition of implication from step 6
8	$p \rightarrow \neg q$	Hypothetical syllogism of steps 4 and 7

Part II: Constructing proofs and mathematical theorems

2. (45 points total) Prove or disprove each of the following statements. Clearly state which proof method you are using.

a. (15 points) There are three consecutive odd integers that are primes, that is, odd primes of the form $p, p+2, p+4$.

Answer:

Proof of an existential by example: 3, 5, 7.

Rosen 3.1.25, page 224.

b. (15 points) n^2-1 is composite whenever n is a positive integer greater than 1.

Answer:

Proof by counter example: 2^2-1 = 3, which is not composite.

Rosen 3.1.22, page 224.

c. (15 points) If n is a positive integer such that the sum of its divisors is $n+1$, then n is prime.

Answer:

Indirect proof: show that if n is not prime, then the sum of its divisors is not $n+1$. If n is not prime, then it at least has factors 1, k, and n, where $1 < k < n$, the sum of which is greater than $n+1$.

Rosen 3.1.38, page 224.

Part III: Problem solving

3. (20 points) The value of a 5-card poker hand is inversely related to is rarity. Thus, a more rare hand (such as a royal flush) will beat a less rare hand (such as a pair). In a poker game using only one deck of cards, one person has a full house (a pair of one face value and a triple of another face value) and another person has a regular flush (all cards of the same suit). Who won? And what is the probability of getting each hand? We are assuming that you can't exchange cards (you are dealt the cards once only). You can leave the answer in combination notation.

Answer:

The order doesn't matter for any of these, so it is dealing with all combinations.

Total number of poker hands is $\binom{52}{5} = 2{,}598{,}960$

Full house: Pick one of thirteen face values for the pair, and from that face value, pick two of the four cards. Pick another of the twelve remaining face values for the three of a kind, and from that face value, pick three of the four cards. The answer is thus $\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{3} = 3744$. The

probability is $\dfrac{\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{3}}{\binom{52}{5}} = \dfrac{3744}{2{,}598{,}960} = 0.00144$.

Flush: Pick one of four suits, then from the thirteen cards of that suit, pick five of them. The answer

is thus $\binom{4}{1}\binom{13}{5} = 5148$. The probability is thus $\dfrac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}} = \dfrac{5148}{2{,}598{,}960} = 0.00198$. Note that if one

did not consider a royal flush (4 possibilities) or a straight flush (36 possibilities) as a regular flush, then the answer would be $5148 - 36 - 4 = 5108$ ($p = \dfrac{5108}{2{,}598{,}960} = 0.00197$), which is what most

poker guides will list. Either flush answer will get full credit.

There are less possibilities (thus a lower probability) for a full house, so the full house beats the flush.

Part IV: Discrete data structures

4. (15 points total) Define each of the following in TWENTY words or less.

a. (5 points) Recursion.

Answer:
See recursion

b. (5 points) Onto (in other words, what the onto property means for a function).

Answer:
A function in which every element of the co-domain has a value mapped onto it.

c. (5 points) One-to-one (in other words, what the one-to-one property means for a function).

Answer:
A function in which each element in the domain maps to a single element in the co-domain.

5. (15 points total) The answers for the following questions should be in the same form as the question (i.e. a graph for part (a), a matrix for part (b), etc.)

a. (5 points) Add the minimum number of edges required to make the following relation an equivalence relation. You may only remove a edge if it prevents the relation from becoming an equivalence relation.
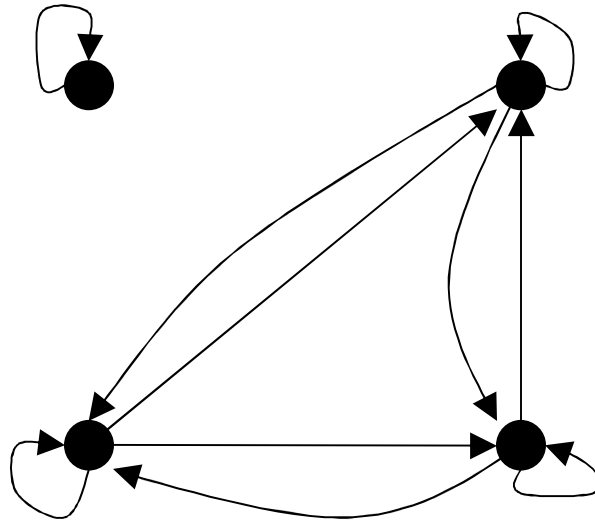
b. (5 points) Consider the relation represented by the following matrix. Add or remove the minimum number of entries to make this relation a partial order. You may only remove an entry if it prevents the relation from becoming a partial order.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} a & 1 & b & 0 & 0 \\ 0 & a & 1 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & a & c \\ 0 & 0 & 0 & c & a \end{bmatrix}$$

All the a's should be 1 (reflexivity); the b should be 1 (transitivity), and only one of the c's should be 1 (antisymmetry). All other values should be zero.

c. (5 points) Consider the relation R on the set S where $S = \{1, 2, 3, 4, 5\}$ and $R = \{ (1,2), (2,3), (3,4), (5,2) \}$. Find the transitive closure of R.

Answer:

In addition to the original edges, the follow edges should be included: (1,3), (1,4), (2,4), (5,3), (5,4)
Thus, the full transitive closure is $R = \{ (1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (5,2), (5,3), (5,4) \}$

Part V: Solving one problem in multiple ways

6. (35 points total) Assume that you only have 5 cent and 7 cent stamps.

a. (5 points) What is the minimum value of postage for which you can create that amount and all greater amounts of postage? In other words, find n such that you can create n or greater amounts of postage using the provided stamps.

Answer:

$n = 24$.

Johnsonbaugh, 1.8.1, page 69.

b. (15 points) Prove this using weak mathematical induction. Clearly label the three steps.

Answer:

Base step: $n = 24$. We can do this by two 5 cent stamps and two 7 cent stamps.

Inductive hypothesis: Assume you can create k cents worth of postage using only 5 and 7 cent stamps.

Inductive step: Show you can create $k+1$ cents of postage using only 5 and 7 cent stamps.

There are two cases for our k cents of postage: either it contains two (or more) 7 cent stamps, or it does not. If it does, replace the two 7 cent stamps (14 cents) with three 5 cent stamps (15 cents), creating $k+1$ cents worth of postage. If it does not, then it contains one or zero 7 cent stamps. In either of these cases, it will also contain four 5 cent stamps (if it contains zero 7 cent stamps, then, as the values are ≥ 24, it must contain at least four 5 cent stamps; if there is one 7 cent stamp, the lowest amount this can be is 27 (as 22 is not being considered since $n \geq 24$)). Thus, we replace the four 5 cent stamps (20 cents) with three 7 cent stamps (21 cents), creating $k+1$ cents of postage.

c. (15 points) Prove this using strong mathematical induction. Clearly label the three steps. This answer must use strong induction.

Answer:

Base steps: $n = 24$. We can do this by two 5 cent stamps and two 7 cent stamps.
                   $n = 25$. We can do this by five 5 cent stamps.
                   $n = 26$. We can do this by one 5 cent stamp and three 7 cent stamps.
                   $n = 27$. We can do this by four 5 cent stamps and one 7 cent stamp.
                   $n = 28$. We can do this by four 7 cent stamps.

Inductive hypothesis: Assume you can create anywhere from 24 to k cents worth of postage using only 5 and 7 cent stamps.

Inductive step: Show you can create $k+1$ cents of postage using only 5 and 7 cent stamps.

To create $k+1$ cents of postage, we will add a 5 cent stamp to $k+1-5$, or $k-4$.

Part VI: The End

7. (5 points) Find four numbers congruent to 5 modulo 17.

   Answer:

   …, -46, -29, -12, 5, 22, 39, 56, …

   (Rosen, chapter 3 supplementary exercises, question 19)

8. (5 points) What your answer is to the following question will not affect your grade; as long as it is answered, you will get full credit for this question. If you feel uncomfortable answering it (it's obviously not anonymous because it is stapled to the rest of your test), please circle N/A, and you will still get full credit for this question.

How helpful were the following parts of the course in terms of helping you *understand the course material*

	Very Unhelpful	Unhelpful	Neutral	Helpful	Very Helpful	Not Applicable
Giving the in-class presentations	1	2	3	4	5	N/A
Listening to the in-class presentations	1	2	3	4	5	N/A