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Pledge: ____

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Signature: ____

There are 180 minutes (3 hours) for this exam and 180 points on the test; don't spend too long on any one question! There is an exam reference sheet as the last page of this exam – feel free to tear it off. You may use scrap paper. However, all answers must be on these twelve exam pages (nothing on the exam reference sheet will be graded).

Part I: Logic	_____/20
Part II: Proofs	_____/50
Part III: Counting	_____/50
Part IV: Structures	_____/55
Part V: Applications	_____/10
Total	_____/180

Part I: Logic

1. (5 points) What is the converse and inverse of $p \rightarrow q$? Clearly label which is which!

Answer:

The converse is $q \rightarrow p$, and the inverse is $\neg p \rightarrow \neg q$.

2. (15 points) Using logical equivalences, prove that $(\neg p \wedge q) \rightarrow (\neg(q \rightarrow p))$ is a tautology. In other words, show that $(\neg p \wedge q) \rightarrow (\neg(q \rightarrow p)) \equiv T$. You must clearly label each step! (Malik/Sen, page 39)

Answer:

$T \equiv (\neg p \wedge q) \rightarrow (\neg(q \rightarrow p))$	Original statement
$T \equiv \neg(\neg p \wedge q) \vee (\neg(\neg q \vee p))$	Definition of implication (twice)
$T \equiv (p \vee \neg q) \vee (q \wedge \neg p)$	DeMorgan's law (twice)
$T \equiv (p \vee \neg q) \vee \neg(\neg q \vee p)$	DeMorgan's law (again)
$T \equiv (p \vee \neg q) \vee \neg(p \vee \neg q)$	DeMorgan's law (again)
$T \equiv T$	Complement law

Note that the last step is because anything or'ed with its complement (here it was $(p \vee \neg q)$ or'ed with its complement) is a tautology.

Part II: Proofs

3. (5 points) Give an English statement that can be proven trivially (i.e. by a trivial proof).

Answer:

Any implication where the consequence is always true.

4. (5 points) Give an English statement that can be proven vacuously (i.e. by a vacuous proof).

Answer:

Any implication where the antecedent is always false.

5. (10 points) Prove, via a proof by contradiction, that if n is an integer, and $3n$+2 is even, then n is even. (Rosen, section 1.5, question 22)

Answer:

Rewrite as a proposition: if $3n$+2 is even, then n is even. Let p be $3n$+2 is even, and q be n is even. Assume that the implication $p \to q$ is false, which only occurs when p is true and q is false. Thus, we are assuming that $3n$+2 is even and that n is odd. If n is odd, then n=2k+1 for some integer k (definition of even numbers). Then $3n+2=3(2k+1)+2=6k+5=2(3k+2)+1$. Thus, since $3n$+2 is 2 times some integer plus one (that integer being $3k$+2), it cannot be even, which contradicts our original assumption (that $3n$+2 is even). Thus, $3n$+2 must be even.

6. (15 points) Given the three propositions $\neg p$, $(\neg q \vee p)$, $\neg r \vee q$, can we conclude r? Show this by using rules of inference. You must label all your steps! (Malik/Sen, page 51, exercise 5)

Answer:

1.	$\neg p$	First hypothesis
2.	$\neg q \vee p$	Second hypothesis
3.	$\neg q$	Disjunctive syllogism on steps 1 and 2
4.	$\neg r \vee q$	Third hypothesis
5.	$\neg r$	Disjunctive syllogism on steps 3 and 4

Therefore, we cannot conclude that r is true (in fact, we conclude that r is false). Note that there are other ways to conclude $\neg r$ from the given hypothesis.

7. (15 points) Using weak mathematical induction, show that $1+2+2^2+\cdots+2^n = 2^{n+1}-1$ for all $n \geq 0$. You must label all your steps! (Malik/Sen, page 143, exercise 2)

Answer:

Let $P(n)$: $1+2+2^2+\cdots+2^n = 2^{n+1}-1$.

Base case:
$$2^0 = 2^{0+1}-1$$
$$1 = 1$$

Inductive hypothesis: $1+2+2^2+\cdots+2^k = 2^{k+1}-1$

Inductive step: $1+2+2^2+\cdots+2^k+2^{k+1} = 2^{k+1+1}-1$

We can replace the $1+2+2^2+\cdots+2^k$ part of the inductive step with the right side of the inductive hypothesis, namely $2^{k+1}-1$, to yield:

$$2^{k+1}-1+2^{k+1} = 2^{k+1+1}-1$$
$$2(2^{k+1})-1 = 2^{k+2}-1$$
$$2^{k+2}-1 = 2^{k+2}-1$$

Part III: Counting

8. (5 points) How many functions are there from the set {1, 2, …, n}, where n is a positive integer, to the set {0, 1}? (Rosen, section 4.1, question 32)

Answer:

For each of the n elements in the domain, the function can map it to either of the 2 values. Thus, there are 2^n total possible functions.

9. (5 points) What is the coefficient of x^7 in $(2x+1)^{13}$? Leave your answer in combinatorial form. (variant of Rosen, section 4.4, question 6)

Answer:

The coefficient is $\binom{13}{7}2^7 = 1716 * 128 = 219{,}648$

10. (5 points) What must be shown in a combinatorial proof? In other words, what must you do in order to prove a formula via a combinatorial proof?

Answer:

You must show that both sides of the equation manage to count the same thing.

11. (15 points) What is the probability of being dealt a straight in a 5-card poker hand? Leave your answer in combinatorial form. Recall that a straight is a series of 5 cards in a sequence. For example, A, 2, 3, 4, 5 is a straight, as is 10, J, Q, K, A (note that the ace can be high or low). Suit does not matter in a straight (and we are ignoring straight flushes and royal flushes).

Answer:

Total number of 5-card stud poker hands is $\binom{52}{5}$.

To pick a straight, we first pick the lowest number, of which there are 10 choices (A through 10). We then pick the suit for each of the 5 cards. This yields $\binom{10}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}$.

Thus, the total probability is $\dfrac{\binom{10}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}} = \dfrac{10240}{2{,}598{,}960} = 0.00394$.

12. (10 points) Let $X = \{x_1, x_2, \ldots, x_{100}\}$ be a set of 100 distinct positive integers. If these positive integers are divided by 75, then show that at least two of the remainders must be the same. (Malik/Sen, p. 435 exercise 2)

Answer:

Proof via pigeonhole principle. The number of pigeons, *N*, is the number of integers (100). The number of pigeonholes, *k*, is the number of possible remainders, or 75. We need to fit 100 numbers into 75 possible remainders (or 100 pigeons into 75 pigeonholes). By the pigeonhole principle, there must be at least one pigeonhole (remainder) that has $\left\lceil \frac{N}{k} \right\rceil = \left\lceil \frac{100}{75} \right\rceil = 2$ pigeons (numbers) in it.

13. (10 points) From the set of integers in the set {1, 2, …, 30}, what is the least number of integers that must be chosen so that at least one of them is divisible by either 3 or 5? Explain your answer. (Malik/Sen, p. 435 exercise 3)

Answer:

There are 14 numbers in the range that are divisible by 3 or 5: {3, 5, 6, 9, 10, 12, 15, 18, 20, 21, 24, 25, 27, 30}. Thus, there are 16 numbers that are not divisible by 3 or 5: {1, 2, 4, 7, 8, 11, 13, 14, 16, 17, 19, 22, 23, 26, 28, 29}. The least number that must be chosen is 1 more than 16 (worst case is that you pick all 16 first, then one that is divisible by 3 or 5). Thus, the answer is 17.

Part IV: Structures

14. (5 points) What is the cardinality of a power set of a set of *n* elements? (déjà vu…)

Answer:

2^n

15. (5 points) Describe, in English, what 1-to-1 and onto mean for functions.

Answer:

1-to-1 means that for every element in the co-domain, there will be one (and only one) element in the domain that maps to it. Onto means that every element in the co-domain has something mapped to it.

16. (5 points) What is the difference between a sequence that has an arithmetic progression, and one that has a geometric progression?

Answer:

An arithmetic progression means each term is a constant amount greater (or less than) the last term. A geometric progression means that each term is a constant factor greater (or less than) the last term.

17. (5 points) Describe, in English, what transitive closure means.

Answer:

Transitive closure means that when there is a path between any two nodes a and b, then there is an edge from a to b.

18. (5 points) What is the difference between asymmetry and antisymmetry?

Answer:

Irreflexivity. An asymmetric relation (such as $<$) must be irreflexive, as any element cannot be related to itself. An antisymmetric relation (such as $\leq$) is allowed to have elements related to themselves, so it does not have to be irreflexive.

19. (5 points) Which properties are required for an equivalence relation and which are required for a partial ordering?

Answer:

Both must be reflexive and transitive. An equivalence relation must be symmetric, and a partial ordering must be antisymmetric.

20. (10 points) Given a relation *R*, what is the difference between R^*, R^{-1}, and $\overline{R}$? Clearly describe what each means.

Answer:

R^* is the transitive closure of *R* (if there is a path between *a* and *b*, then there is an edge between *a* and *b*). R^{-1} is the inverse relation, where all the edges are reversed (formally, $R^{-1} = \{(b,a) \mid (a,b) \in R\}$). $\overline{R}$ is the complementary relation, which contains all the edges that are not in R (formally, $R^{-1} = \{(a,b) \mid (a,b) \notin R\}$).

21. (15 points) Given sets A and B, prove that $(A-B)\bigcup(B-A)=(A\bigcup B)-(B\bigcap A)$. You can either use set builder notation or set equivalences, but you cannot use membership tables. (variant of Malik/Sen, page 22, exercise 7)

Answer:

$(A-B)\bigcup(B-A)=(A\bigcup B)-(B\bigcap A)$	Original statement
$=(A\bigcup B)\bigcap\overline{(B\bigcap A)}$	Definition of difference
$=(A\bigcup B)\bigcap(\overline{B}\bigcup\overline{A})$	DeMorgan's law
$=\left((A\bigcup B)\cap\overline{B}\right)\cup\left((A\bigcup B)\cap\overline{A}\right)$	Distributive law
$=\left((A\cap\overline{B})\cup(B\cap\overline{B})\right)\cup\left((A\cap\overline{A})\cup(B\cap\overline{A})\right)$	Distributive law (again)
$=\left((A\cap\overline{B})\cup\varnothing\right)\cup\left(\varnothing\cup(B\cap\overline{A})\right)$	Complement law
$=\left(A\cap\overline{B}\right)\cup\left(B\cap\overline{A}\right)$	Identity law
$=(A-B)\bigcup(B-A)$	Definition of difference

Part V: Applications

22. (10 points) State a real world application of each of the following discrete mathematical concepts.

- Boolean logic

Answer:
CPUs (arithmetic applications are done via Boolean logic)

- Mathematical induction

Answer:
Verifying program correctness.

- Relations

Answer:
Relational databases or MapQuest.

- Prime numbers

Answer:
The RSA algorithm, or encryption.

- Algorithms

Answer:
Computer programs!

- Fibonacci sequence

Answer:
Reproducing rabbits, conch shell radii, etc.

CS/APMA 202

Final Exam Reference Sheet

Set and logical identities

Sets (Rosen, p. 89)	Name	Boolean logic (Rosen, p. 24)
$A \cup \varnothing = A$ $A \cap U = A$	Identity laws	$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$
$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination laws	$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$
$A \cup A = A$ $A \cap A = A$	Idempotent laws	$p \vee p \equiv p$ $p \wedge p \equiv p$
$\left(\overline{\overline{A}}\right) = A$	Complementation law	$\neg(\neg p) \equiv p$
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws	$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	DeMorgan's laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$
$A \cup \overline{A} = U$ $A \cap \overline{A} = \varnothing$	Complement laws	$p \vee \neg p = \mathbf{T}$ $p \wedge \neg p = \mathbf{F}$

Rules of inference (Rosen, p. 58)

Rule of Inference	Tautology	Name
$\underline{p}$ $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$\underline{p \wedge q}$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p $\underline{q}$ $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
p $\underline{p \to q}$ $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus ponens
$\neg q$ $\underline{p \to q}$ $\therefore \neg p$	$[\neg q \wedge (p \to q)] \to \neg p$	Modus tollens
$p \to q$ $\underline{q \to r}$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\underline{\neg p}$ $\therefore q$	$[(p \vee q) \wedge \neg p] \to q$	Disjunctive syllogism
$p \vee q$ $\underline{\neg p \vee r}$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$	Resolution

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Do not include any work on this page.
It will not be graded