Union-Closed Systems of Sets and the Frankl Conjecture

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Abstract

The "union-closed" conjecture, sometimes attributed to Peter Frankl, dates from 1979. It asserts that in any finite, union-closed system, $\mathcal{S}$, of sets, there is some element x that is a member of at least half the sets in $\mathcal{S}$. It seems simple, but to date no one has been able to either prove or disprove it. In this article we prove it to be true.

To do this each system of sets must be assigned to one of two classes depending on whether the intersection of its largest sets is empty or not. Then two different proof techniques will be required.

A system of sets, $\mathcal{S}$, is said to be **union-closed** if $X, Y \in \mathcal{S}$ implies $Z = X \cup Y \in \mathcal{S}$. All sets, X, Y, Z are sets over some finite universe U of elements. In the remainder of this paper, $\mathcal{S}$, which we will refer to as a U-C system, is always assumed to be union-closed.

Conjecture: *Given a non-empty, finite, union-closed system, $\mathcal{S}$, over a set of elements, U, there exists some element $x \in U$ such that x is an element in (member of) at least half the sets of $\mathcal{S}$.*

This conjecture, which has been attributed to Peter Frankl in 1979, appears to be absurdly simple. Yet, in 2015 Bruhn and Schaudt observe that "despite its apparent simplicity the union-closed sets conjecture remains wide open. This is certainly not for lack of interest — there are about 50 articles dedicated to the conjecture as well as several websites" [1].

This conjecture can be rewritten as: there exists $X \subseteq U$ such that

$$\{Y \in \mathcal{S} : X \subseteq Y\}| \geq |\{Y \in \mathcal{S} : X \not\subseteq Y\}|. \tag{1}$$

since we can always consider any element x to be a singleton set $X = \{x\}$. X need not be a set in $\mathcal{S}$, and seldom is. A set X is said to be **abundant** if it satisfies the inequality (1). Bruhn and Schaudt explain that "we do not know where to expect an abundant element" [1]. In this paper we show that one looks for an abundant set, or element, in different places depending on the structure of $\mathcal{S}$. In

Section 2, Lemmas 1.2 and 2.1 culimnate in Proposition 2.2 to establish that it is the complement of one of the largest sets that must be abundant. In Section 3, Proposition 3.4 details where among the smallest sets one can find an abundant set.

This conjecture can be regarded as the "holy grail" of union-closed (U-C) systems of sets.

1 Systems of Sets Closed under Union

A set Z is said to **cover** a set X with respect to $\mathcal{S}$ if $X \subset Z$ but $X \subseteq Y \subseteq Z$, where $Y \in \mathcal{S}$, implies either $X = Y$ or $Y = Z$, *i.e.* there can be no set $Y \in \mathcal{S}$ "between" X and Z. If a set $Y \in \mathcal{S}$ covers the empty set, $\emptyset$, we call Y an **atom** of $\mathcal{S}$ and denote it by A_i. The collection of all sets $\{A_i\}$ covering $\emptyset$, we denote by $\mathcal{A}$. Conversely, those sets Z_i covered by U, we call **co-atoms** and let $\mathcal{Z}$ denote this collection $\{Z_i\}$.

By $\bar{X}$ we mean $U \sim X$, or the **complement** of X. The dual nature of complements, such as (a) $X \subseteq Y$ if and only if $\bar{Y} \subseteq \bar{X}$, and (b) $Z = X \cup Y$ if and only if $\bar{Z} = \bar{X} \cap \bar{Y}$, is well known and easily demonstrated.

A non-empty set Z in an union-closed system $\mathcal{S}$ is said to be **reducible**, if it is the union of two distinct, non-empty sets of $\mathcal{S}$. It is **irreducible** if there do not exist two such sets, in which case there are only two possible situations:

(1) Z covers $\emptyset$ (so Z is an atom); or

(2) $Z = X \cup Y$ where $X \in \mathcal{S}$ and $Y \notin \mathcal{S}$.

In the latter case, we assume Z consists of a set $X \in \mathcal{S}$ and a set $[Y] = Z \sim X$, where $[Y] \notin \mathcal{S}$.[1] Readily, a set Z, that is not an atom, is irreducible if and only if it covers a single set $Y \in \mathcal{S}$.

Lemma 1.1 *Let $\mathcal{S}$ be a U-C system with $Y \in \mathcal{S}$. $\mathcal{S}' = \mathcal{S} \sim Y$ is a U-C system if an only if Y is irreducible.*

Proof: If Y is reducible, $Y = X \cup V$ where $X, V \in \mathcal{S}$ implying $\mathcal{S} \sim Y$ will not be union closed; so Y must be irreducible. Conversely, any irreducible set can be deleted from a U-C system to yield another U-C system because it covers at most a single set. □

It is curious that it is essentially trivial to obtain another U-C system, $\mathcal{S}'$, by removing only one irreducible set; but it is extremely difficult to add just a single set and still be union-closed.

By an **interval** in $\mathcal{S}$ we mean $\langle X, Z \rangle = \{Y \in \mathcal{S} : X \subset Y \subset Z\}$. Note that the bounding sets X and Z may, or may not, be in $\mathcal{S}$.

[1] We use square brackets, $[\ldots]$ as in $X \cup [Y]$, to signal that the enclosed set is not a set of $\mathcal{S}$. Similarly, we use $[x]$ when $\{x\} \notin \mathcal{S}$.

1.1 Co-atoms

Let Z_k be a co-atom, and let $\bar{Z}_k = U{\sim}Z_k$. It is not difficult to show that for every co-atom, $\bar{Z}_k$ is either a single element, or a block[2] that can be replaced by a single element. Clearly, for all $Y \in \langle\emptyset, Z_k\rangle$, $\bar{Z}_k \not\subseteq Y$ since $\bar{Z}_k \not\subseteq Z_k$.

Lemma 1.2 *Let $Y \in \mathcal{S}$. If $Y \not\subseteq Z_k$ then $\bar{Z}_k \subseteq Y$. Or equivalently, $Y \notin \langle\emptyset, Z_k\rangle$ then $Y \in \langle\emptyset, \bar{Z}_k\rangle$.*

Proof: $Y \not\subseteq Z_k$ implies $Z_k \subseteq Y \cup Z_k \subseteq U$. But U covers Z_k, so either $Z_k = Y \cup Z_k$ or $Y \cup Z_k = U$. Since the former is impossible by assumption, $\bar{Z}_k \subseteq Y$. □

This is a rather surprising lemma and worth examining further. In Figure 1, where the co-atoms are numbered left to right and their complements, $\bar{Z}_i$ indicated as superscripts, $-k$. We have delimited $\langle\emptyset, Z_3\rangle$ = $\langle\emptyset, \{12356\}\rangle$ with bolder lines and a dashed enclosure. $\bar{Z}_3 = [4]$ and we observe

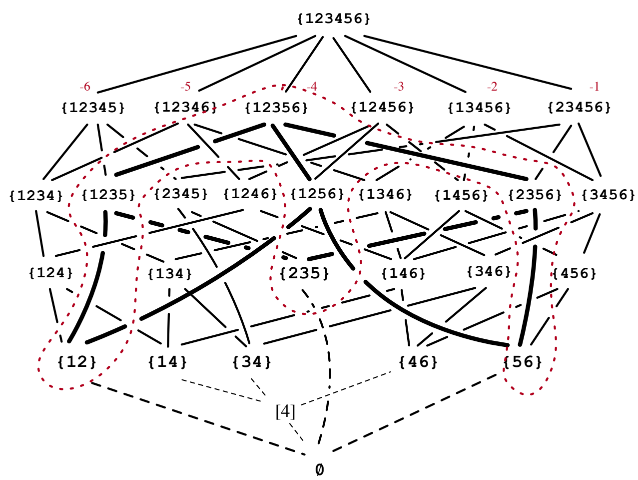

Figure 1: $\langle\emptyset, \{12356\}\rangle$, $\bar{Z}_3 = [4]$.

that for every non-empty set $Y \notin \langle\emptyset, Z_3\rangle$, $[4] \subseteq Y$, even though $[4]$ is not a set of $\mathcal{S}$.

Another example involves Poonen's classic U-C system [3], shown as Figure 2, which was created to provide a counter-example to the rather natural supposition that if $|A_i|$ were minimal, then the members of A_i would also be members of many other sets Y.

Poonen's U-C system [3] shown as Figure 2, is one counter-example of the supposition. Readily, $\{123\}$ is the minimal atom; but for all $x \in \{123\}$, $|\{Y : [x] \subseteq Y\}| = 10 < 13 = 1/2 * 26 = |\mathcal{S}|/2$. We use this U-C sysem to provide a second illustration of Proposition 1.2. In Figure 2 we have delimited $\langle\emptyset, Z_4\rangle$, which is not minimal, with a dashed line and enboldened their connections.

[2] A block is a set of elements that always appear together.

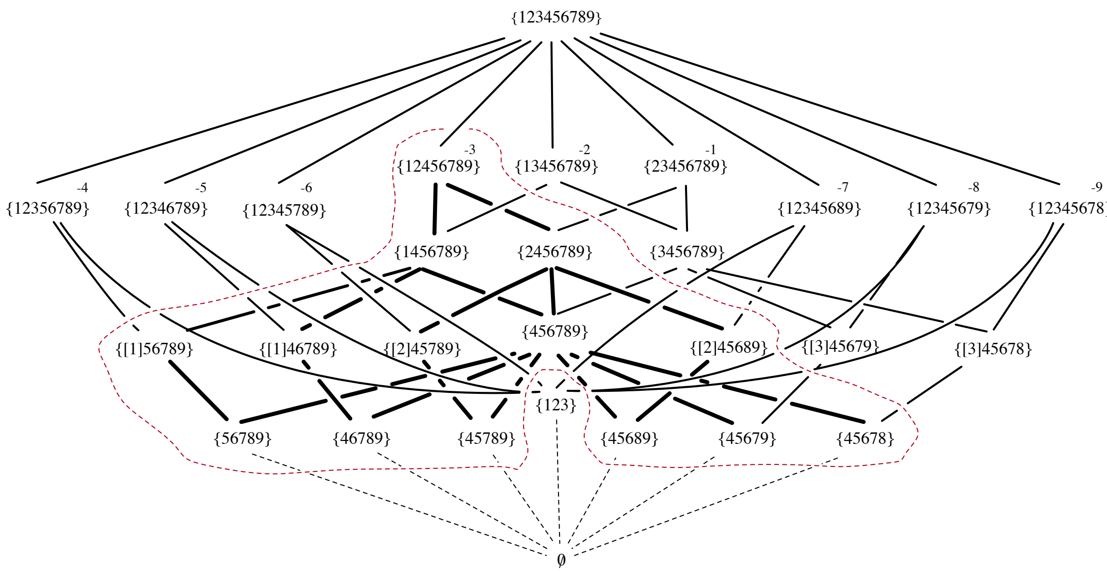

Figure 2: The U-C system of Poonen [3].

Observe that for all non-empty $Y \notin \langle \emptyset, Z_4 \rangle$, $[3] = \bar{Z}_4 \subseteq Y$. In the U-C systems illustrated in Figures 1 or 2, neither $\bar{Z}_3 = [4]$ in the former nor $\bar{Z}_4 = [3]$ of the latter is a subset of $\mathcal{S}$.

It is easy to show that any singleton set $\{a\} \in \mathcal{S}$ is "abundant" since for all $Y \in \mathcal{S}$ such that $\{a\} \not\subseteq Y$, there exists $\{a\} \cup Y \in \mathcal{S}$. This can be extended to show that any system, $\mathcal{S}$, with a doubleton atom, $\{ab\}$ is also "abundant". Bruhn and Schaudt [1] observe that trying to extend this approach to tripleton atoms, $\{abc\}$ fails. So, we are only concerned with U-C systems for which $Y \in \mathcal{S}$ implies $|Y| \geq 3$. Consequently, we can assume that an aundant set X is of the form $[x]$ since $|X| > 2$ and every element $x \in X$ is abundant.

Before continuing, we can eliminate two trivial kinds of U-C systems.

Lemma 1.3 *If* $|\mathcal{A}| = 1$ *or* $|\mathcal{Z}| = 1$ *then there exists* $[x]$ *such that* $|\{Y \in \mathcal{S} : [x] \subseteq Y\}| \geq |\{Y \in \mathcal{S} : [x] \not\subseteq Y\}|$.

Proof: If $|\mathcal{A}| = 1$, with $A_1 \in \mathcal{A}$, then for all $Y \in \mathcal{S}$, $A_1 \subseteq Z$, so the inequality is trivial.
If $|\mathcal{Z}| = 1$, let $Z_1 \in \mathcal{Z}$. We may inductively assume that the inequality is true for $\mathcal{S}' = \langle \emptyset, Z_1 \rangle$ where $U' = Z_1$. The satisfying set $[x'] \subseteq U$, so again the inequality is trivial. □

The proof technique of Lemma 1.3 employs an induction on the cardinality, $|\mathcal{S}|$. Readily, if $|\mathcal{S}| = 1$ the conjecture (1) is satisfied. We can that assume that if $\mathcal{S}' \subset \mathcal{S}$ then $\mathcal{S}'$ satisfies (1) *provided* $\mathcal{S}'$ is U-C. Lemma 1.1 will play its role here.

2 When $\bigcap_j \{Z_j \in \mathcal{Z}\} = \emptyset$

In this section we explore those U-C systems for which $\cap_j \{Z_j \in \mathcal{Z}\} = \emptyset$. Since $\cap_j \{Z_j \in \mathcal{Z}\} = U \sim \cup_j \{\bar{Z}_j \in \mathcal{Z}\}$ we will assume that each $\bar{Z}_i$ denotes a single element of U.[3] The U-C systems of Figures 1 and 2 are members of this class.

Proposition 1.2 identifies one source of many abundant sets, namely the complements of co-atoms, $\bar{Z}_k$. This will lead to Proposition 2.2 which provides a partial answer to the U-C conjecture. But, first we must establish that if $|\langle\emptyset, Z_k\rangle|$ is minimal then $|\langle\emptyset, Z_k\rangle| \leq |\mathcal{S}|/2$, provided $\cap_j \{Z_j \in \mathcal{Z}\} = \emptyset$. This seems intuitively obvious, but the concluding proviso is essential as shown by the ssimple counter-example of Figure 3. Here $|\langle\emptyset, Z_1\rangle| = |\langle\emptyset, Z_2\rangle| = 3$, so both are minimal. But,

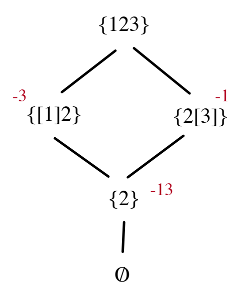

Figure 3: $\cap_j \{Z_j \in \mathcal{Z}\} = \{2\} \neq \emptyset$.

$|\langle\emptyset, Z_1\rangle| = 3 > 5/2 = |\mathcal{S}|/2$.

Lemma 2.1 *Let $\cap_j \{Z_j \in \mathcal{Z}\} = \emptyset$. If $|\langle\emptyset, Z_k\rangle|$ is minimal, then $|\langle\emptyset, Z_k\rangle| \leq |\mathcal{S}\sim\langle\emptyset, Z_k\rangle|$, with equality if and only if for each $Y_i \in \langle\emptyset, Z_k\rangle$, we have $\bar{Y}_i \in \mathcal{S}\sim\langle\emptyset, Z_k\rangle$.*

Proof: We run a finite induction on $|Y_i \in \langle\emptyset, Z_k\rangle|$. If $|Y_i \in \langle\emptyset, Z_k\rangle| = 0$ then $|\mathcal{Z}| \geq 2$ ensures that $|CALS| \geq 4$ and $|\langle\emptyset, Z_k\rangle| = 2 \leq |\mathcal{S}|/2$.

Let $|\langle\emptyset, Z_k\rangle| = 1$ with $Y_1 \in \langle\emptyset, Z_k\rangle$. Since $|\langle\emptyset, Z_k\rangle|$ is minimal, for all $j \neq k$, $|Y_i \in \langle\emptyset, Z_j\rangle| \geq 1$. Suppose $|\mathcal{S}\sim\langle\emptyset, Z_k\rangle| = 0 < 1$, then $Y_1 \in \langle\emptyset, Z_j\rangle$ for all j, contradicting the assumption that $\cap_j \{Z_j \in \mathcal{Z}\} = 0$. So $|\mathcal{S}\sim\langle\emptyset, Z_k\rangle| > 0$. If for $j \neq k$, $Y_1 \notin \langle\emptyset, Z_j\rangle$ then $\exists X_1 \neq Y_1$ where $X_1 \in \langle\emptyset, Z_j\rangle$ implying $X_1 \in \mathcal{S}\sim\langle\emptyset, Z_k\rangle$ and $|\langle\emptyset, Z_k\rangle| \leq |\mathcal{S}\sim\langle\emptyset, Z_k\rangle|$. Finally, if $|\mathcal{S}\sim\langle\emptyset, Z_k\rangle| = 1$, then $X_1 \in \langle\emptyset, Z_j\rangle$ for all $j \neq k$ then by Lemma 1.2 $X_1 = \bar{Z}_k$.

Assume the lemma is true for $|Y_i \in \langle\emptyset, Z_k\rangle| < m$ and let $|Y_i \in \langle\emptyset, Z_k\rangle| = m$, with $Y_m \in \langle\emptyset, Z_k\rangle$. By inductive assumption, $\mathcal{S}\sim\langle\emptyset, Z_k\rangle| \geq m - 1$. If $|\mathcal{S}\sim\langle\emptyset, Z_k\rangle| = m - 1$ then the argument above implies $Y_m \in \cap_j \{Z_j \in \mathcal{Z}\}$ and the same contradiction. Thus $|\mathcal{S}\sim\langle\emptyset, Z_k\rangle| \geq m$ with equality implying $\bar{Y}_m \in \langle\emptyset, Z_j\rangle$ for all $j \neq k$. $\Box$

[3]This class corresponds to the "first class" of FC(n) systems explored by Vaughan [5], that is, the n-set U, together with all of its $(n-1)$-subsets.

Proposition 2.2 *Let* $|\mathcal{Z}| = |U| \geq 2$ *and let* $\cap_j \{Z_j \in \mathcal{Z}\} = \emptyset$. *If* $|\langle \emptyset, Z_k \rangle|$ *is minimal then* $|\{Y \in \mathcal{S} : \bar{Z}_k \subseteq Y\}| \geq |\{Y \in \mathcal{S} : \bar{Z}_k \not\subseteq Y\}|$.

Proof: By Lemmas 1.2 and 2.1, $|\{Y \in \mathcal{S} : \bar{Z}_k \not\subseteq Y\}| = |\langle \emptyset, Z_k \rangle| \leq |\mathcal{S} \sim \langle \emptyset, Z_k \rangle| = |\{Y \in \mathcal{S} : Z_k \subseteq Y\}|$. Reverse this inequality to obtain (1). □

Consequently, this result resolves Frankl's conjecture for this large class of U-C systems. $\bar{Z}_k = [x]$ an abundant set for any co-atom, Z_k, such that $|\langle \emptyset, Z_k \rangle|$ is minimal.

3 When $\bigcap_j \{Z_j \in \mathcal{Z}\} = I \neq \emptyset$

In this section we are concerned with the structure of $\mathcal{S}$ when the intersection, I, of all the co-atoms $Z_j \in \mathcal{Z}$ is non-empty. The resulting set, I, may be a member of $\mathcal{S}$, or not. A completely different proof structure will be required.

Figure 4 illustrates two representative U-C systems for which $I = \cap_j \{Z_j \in \mathcal{Z}\} \neq \emptyset$. In Figure

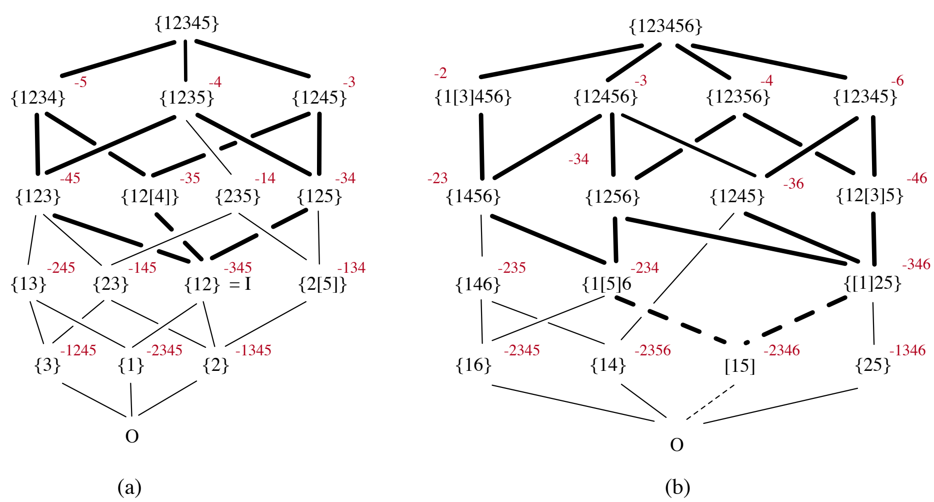

Figure 4: Two U-C systems in which $I = \cap_j \{Z_j \in \mathcal{Z}\} \neq \emptyset$.

4(a), $I = \{12\} = \{12345\} \sim \{345\} = U \sim \cup_j \{\bar{Z}_j\} \in \mathcal{S}$. In Figure 4(b), $I = [15] \notin \mathcal{S}$. In both, the interval $\langle I, U \rangle$ has been emboldened, and $\mathcal{S} \sim \langle I, U \rangle$ is denoted with thinner, but solid, lines. One expects that an abundant set will be found in I, or some subset of I. In both these figures it is easy to verify that I is itself abundant.

However, this need not always be the case. In Figure 5 , $I = [3]$ is clearly not abundant since $|\{Y \in \mathcal{S} : [3] \subseteq Y\}| = 4 < 7 = |\{Y \in \mathcal{S} : [3] \not\subseteq Y\}|$. Readily $I = [3]$ has no subset that might be abundant.

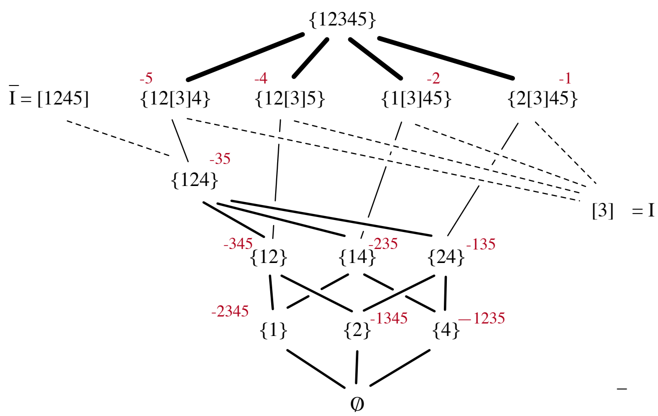

Figure 5: $I = [3]$ is not abundant.

Proposition 3.1 *If* $|\langle I, U\rangle| \geq |\mathcal{S} \sim \langle I, U\rangle|$ *then* $|\{Y \in \mathcal{S} : I \subseteq Y\}| \geq |\{Y \in \mathcal{S} : I \not\subseteq Y\}|$.

Proof: Trivial □

Proposition 3.1 resolves the conjecture whenever $|\langle I, U\rangle| \geq |\mathcal{S}{\sim}\langle I, U\rangle|$ since I itself and any subset $[x] \subset I$ will be abundant. Consequently, we need only be concerned when the number of sets not in $\langle I, U\rangle$ outnumber those in it. In Figures 4(a) and (b), $\mathcal{S} \sim \langle I, U\rangle$ consists of the sets "below" the bolder $\langle I, U\rangle$ that are connected by the thinner lines. It is no longer assured that an abundant set $[x]$ will be contained in I; we may find $[x] \in \mathcal{S}{\sim}\langle I, U\rangle$. The rest of this section is focused on the structure of $\mathcal{S} \sim \langle I, U\rangle$.

Proposition 3.2 *If* $\mathcal{S} \sim \langle I, U\rangle \subseteq \langle \emptyset, I\rangle$ *then there exists* $[x] \subseteq I$ *such that* $|\{Y \in \mathcal{S} : [x] \subseteq Y\}| > |\{Y \in \mathcal{S} : [x] \not\subseteq Y\}|$.

Proof: $\mathcal{S}' = \langle \emptyset, I\rangle$ is a U-C system contained in $\mathcal{S}$. It is assumed that the U-C conjecture is true for $\mathcal{S}'$. Let $[x]$ be that set. Since $[x] \subseteq I$, for all $Y \in \langle I, U\rangle$, $[x] \subseteq Y$ and the result follows easily. □

A key subset of $\mathcal{S} \sim \langle I, U\rangle$ is the interval $\langle \emptyset, \bar{I}\rangle$. $\bar{I} \notin \mathcal{S}$ since otherwise $\bar{I} \in \mathcal{Z}$ and $\cap_j\{Z_j \in \mathcal{Z}\} = \emptyset$, contradicting the assumed co-atom structure. (In fact, no set $Y \in \langle \bar{I}, U\rangle$ can be in $\mathcal{S}$ for the same reason.) $\langle \emptyset, \bar{I}\rangle$ in Figure 5 is interesting. $\bar{I} = [1245] \notin \mathcal{S}$ and $\langle \emptyset, \bar{I}\rangle = \langle \emptyset, [1245]\rangle$ consists of the eight sets $\{\emptyset, \ldots, \{124\}\}$, so $|\langle \emptyset, \bar{I}\rangle| = 8 > 6 = |\langle [3], U\rangle| = |\langle I, U\rangle|$. Because $\langle \emptyset, \bar{I}\rangle$ is U-C, $\cup_i\{Y_i \in \langle \emptyset, \bar{I}\rangle\} \subset \bar{I}$. Consequently, $\langle \emptyset, \bar{I}\rangle$ must have a unique greatest element which is an element of $\mathcal{S}$, but cannot be $\bar{I}$. We see this in Figure 5 where $\{124\} \subset [1245] = \bar{I}$.

Proposition 3.3 *If* $\langle \emptyset, \bar{I}\rangle$ *is not empty and* $\langle \emptyset, \bar{I}\rangle = \mathcal{S} \sim \langle I, U\rangle$ *then there exists* $[x] \subseteq \bar{I}$ *satisfying the inequality* (**??**).

Proof: Since $\langle \emptyset, \bar{I}\rangle$ is a U-C family $|\langle \emptyset, \bar{I}\rangle| < |\mathcal{S}|$ we may assume there exists an abundant $[x] \in A_1$ where $\emptyset \subset A_1 \subset \bar{I}$. Now let $Y_i \in \langle I, U\rangle$, $[x] \not\subseteq Y_i$. Since $\mathcal{S}$ is U-C there exists $X_i = Y_i \cup A_1$ such that $[x] \subseteq X_i$.

Moreover, because I and $\bar{I}$ are disjoint, $Y_i \neq Y_j \in \langle I, U\rangle$ implies $X_i = Y_i \cup A_1 \neq Y_j \cup A_1 = X_j$. Thus $[x]$ is abundant in $\mathcal{S}$. □

Figure 6 graphically illustrates relationships between $\langle I, U\rangle, \mathcal{S} \sim \langle I, U\rangle$, and $\langle \emptyset, \bar{I}\rangle$. Here the

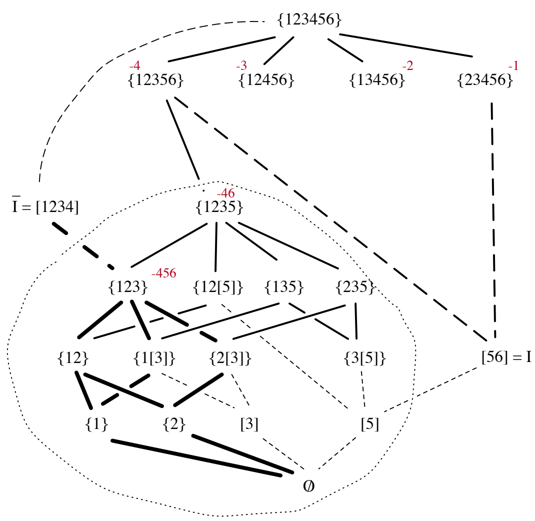

Figure 6: A schematic representation of $\langle \emptyset, \bar{I}\rangle$ contained in $\mathcal{S} \sim \langle I, U\rangle$.

characterising parameters of $\mathcal{S}$ are $|U| = 6, |\mathcal{Z}| = 4$. $I = [56]$ which may, or may not, be a set in $\mathcal{S}$. $\langle I, U\rangle$ is only sketched out with dashed lines; it may consist of other sets, but need not be. But, we assume that $|\langle I, U\rangle| < |\mathcal{S} \sim \langle I, U\rangle|$ else Proposition 3.1 is invoked. Note that if $I = \{56\} \in \mathcal{S}$ there would be at least 11 more sets, $\{156\}, \ldots, \{12356\} \in \langle I, U\rangle$. $\mathcal{S} \sim \langle I, U\rangle$ has been circled by a dotted line and $\langle \emptyset, \bar{I}\rangle$ within it is indicated by bolder lines.

Because of Propositions 3.1, 3.2 and 3.3, the only remaining case involves $\mathcal{S} \sim \langle I, U\rangle$, where $|\mathcal{S}{\sim}\langle I, U\rangle| \geq |\langle I, U\rangle|$ and there exists at least one $Y \in \mathcal{S} \sim \langle I, U\rangle$ such that $Y \cap I \neq \emptyset$, but $Y \not\subseteq I$.

Proposition 3.4 *Let $\bigcap_j \{Z_j \in \mathcal{Z}\} = I \neq \emptyset$ and let there exist $Y \in \mathcal{S} \sim \langle I, U\rangle$ such that $Y \cap I \neq \emptyset$.*
(a) If $|\langle \emptyset, \bar{I}\rangle| \leq |\mathcal{S}|/2$ then there exists $[x] \subseteq I$ such that …
(b) If $|\langle \emptyset, \bar{I}\rangle| \geq |\langle I, U\rangle|$ then there exists $[x] \subseteq \bar{I}$ such that …
$|\{Y \in \mathcal{S} : [x] \subseteq Y\}| \geq |\{Y \in \mathcal{S} : [x] \not\subseteq Y\}|$.

Proof: As in Section 1, we can assume that $Y \in \mathcal{S}$ implies $|Y| > 2$ else the result is trivial. Readily, $|\langle I, U\rangle| + |\mathcal{S} \sim \langle I, U\rangle| = |\mathcal{S}|$, and since $\langle \emptyset, \bar{I}\rangle \subseteq \mathcal{S} \sim \langle I, U\rangle$, $|\langle \emptyset, \bar{I}\rangle| \leq |\mathcal{S} \sim \langle I, U\rangle|$.
Let $I = \{x_1 x_2 \ldots x_t\}$. We will run an induction on $x_i \in I$.
Initially, let $i = 1$ and let $Y \in \mathcal{S} \sim \langle I, U\rangle$.
(a) If $[x_1] \not\subseteq Y$ then $Y \subseteq \bar{I}$. Thus $|\{Y \in \mathcal{S} \sim \langle I, U\rangle : [x_1] \not\subseteq Y\}| = |\langle \emptyset, \bar{I}\rangle| = |\{Y \in \mathcal{S} : [x_1] \not\subseteq Y\}|$, because for all $Y \in \langle I, U\rangle, [x_1] \subset Y$. Consequently, if $|\langle \emptyset, \bar{I}\rangle| \leq |\mathcal{S}|/2 = (|\langle I, U\rangle| + |\mathcal{S} \sim \langle I, U\rangle|)/2$ then

$[x_1] \subseteq I$ is abundant.
(b) Since $\langle \emptyset, \bar{I} \rangle$ is itself a U-C system we can assume $\exists x' \in \bar{I}$ where $[x']$ is an abundant set in $\langle \emptyset, \bar{I} \rangle$. If $Y \in \mathcal{S} \sim \langle I, U \rangle$ and $Y \cap I \neq \emptyset$ then $Y = Y' \cup [x_1]$ for some $Y' \in \langle \emptyset, \bar{I} \rangle$. Thus $[x']$ must be in at least $(|\mathcal{S} \sim \langle I, U \rangle| + |\langle \emptyset, \bar{I} \rangle|)/2$ sets of $\mathcal{S}$. So $[x']$ is abundant if $(|\mathcal{S} \sim \langle I, U \rangle| + |\langle \emptyset, \bar{I} \rangle|)/2 \geq (|\mathcal{S} \sim \langle I, U \rangle| + |\langle I, U \rangle|)/2 = |\mathcal{S}|/2$ or equivalently, if $|\langle \emptyset, \bar{I} \rangle| \geq a$. (Figure 6 illustrates a U-C system with the properties of this initial configuration (b) where $Y = \{1235\}$ and $[x_1] = [5]$. Increasing the number of sets in $\langle I, U \rangle = \langle [56], U \rangle$ would eventually yield condition (a).)

Thus both (a) and (b) are true when $i = 1$; we inductively assume they are true when $i = s < t$. Let $\mathcal{S}$ be any U-C system satisfying the condition that if $Y \cap I \neq \emptyset$ then $Y \cap I \subseteq \{x_1 x_2 \ldots x_t\}$. We construct a smaller U-C system $\mathcal{S}' \subset \mathcal{S}$ as follows. For all Y in either $\langle I, U \rangle$ or $\langle \emptyset, \bar{I} \rangle$, let $Y = Y' \in \mathcal{S}'$. For $Y \in \mathcal{S} \sim \langle I, U \rangle$ (excluding $\langle \emptyset, \bar{I} \rangle$), if $[x_t] \not\subset Y$, again let $Y = Y' \in \mathcal{S}'$, but if $[x_t] \subset Y$ and $Y {\sim} [x_t] \notin \mathcal{S} \sim \langle I, U \rangle$ (that is $Y {\sim} [x_t]$ is not already a set of Y') then $Y {\sim} [x_t] = Y' \in \mathcal{S}'$, otherwise $Y \notin \mathcal{S}'$.

Since $Y \in \mathcal{S}$ is deleted only if $Y {\sim} [x_t] \in \mathcal{S} \sim \langle I, U \rangle$, Y is irreducible, and by Lemma 1.1, $\mathcal{S}'$ so created is U-C and $\mathcal{S}'$ satisfies the conditions of the proposition. Now consider $\mathcal{S}$ as an extension of $\mathcal{S}'$. $|\mathcal{S}'| \leq |\mathcal{S}|$ with equality if no irreducible sets of the form $Y \cup [x_t]$ exist in $\mathcal{S}$. Thus if $|\langle \emptyset, \bar{I} \rangle| \leq |\mathcal{S}'|/2$ in $\mathcal{S}'$, then since $\langle \emptyset, \bar{I} \rangle$ is unchanged in $\mathcal{S}'$, $|\langle \emptyset, \bar{I} \rangle| \leq |\mathcal{S}|/2$ in $\mathcal{S}$ and (a) follows. And since no sets of $\langle \emptyset, \bar{I} \rangle$ or $\langle I, U \rangle \in \mathcal{S}$ are deleted, if $|\langle \emptyset, \bar{I} \rangle| \geq |\langle I, U \rangle|$ in $\mathcal{S}'$ then $|\langle \emptyset, \bar{I} \rangle| \geq |\langle I, U \rangle|$ in $\mathcal{S}$, so conclusion (b) follows. □

All possible configurations, when $\cap_j \{Z_j \in \mathcal{Z}\} = I \neq \emptyset$, have been accounted for. But this final proposition also demonstrates that whenever $|\langle I, U \rangle| \leq |\langle \emptyset, \bar{I} \rangle| \leq |\mathcal{S}|/2$ we are assured that abundant elements $[x_i]$ can be found in *both* I and $\bar{I}$.

4 Epilogue

Either $\cap_j \{Z_j \in \mathcal{Z}\} = \emptyset$, or not. Taken together, Propositions 2.2, 3.1, 3.2 and 3.4 resolve Frankl's U-C Conjecture. It is true — as almost everyone has thought it was.

The other important result of this paper is uncovering the bifurcated nature of U-C systems. They fall into two distinct structural classes, Class I, where $I = \cap_j \{Z_j \in \mathcal{Z}\} = \emptyset$, and Class II, where $I = \cap_j \{Z_j \in \mathcal{Z}\} \neq \emptyset$. Because of this no single proof technique could work for both. In Class II, the atom set $\mathcal{A}$ tends to be small, so one looks for some abundant element $[x]$ in $\mathcal{A}$, frequently $[x] \subseteq I$. In Class I, $\mathcal{A}$ can be rather large, while $|\mathcal{Z}|$ is limited by n, so one searches for an abundant element in $\bar{Z}$.

It appears that the relatively novel concepts of co-atoms, intervals and specifically the roles of $\langle \emptyset, \bar{I} \rangle$ and $\langle I, U \rangle$ will be useful tools in further investigations of these two classes of union-closed set systems. And since the complement of union-closed systems are intersection-closed, these discoveries are applicable to finite closure systems [2] as well.

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References

[1] Henning Bruhn and Oliver Schaudt. The journey of the union-closed sets conjecture. *Graphs Combin.*, 31(6):2043–2074, 2015. doi:/10.1007/s00373-014-1515-0.

[2] John L. Pfaltz and Robert E. Jamison. Closure Systems and their Structure. *Information Sciences*, 139:275–286, 2001.

[3] Bjorn Poonen. Union-Closed Families. *J. of Combinatorial Theory, Series A*, 59:252–268, 1992. doi:10.1016/0097-3165(92)90068-6.

[4] Luca Studer. An Asymptotic Version of Frankl's Conjecture. *Amer. Math Monthly*, 128(7):652–654, Aug-Sept. 2021. doi:/10.1080/00029890.2021.1926192.

[5] Theresa P. Vaughan. Families Implying the Frankl Conjecture. *European J. Combinatorics*, 23:851–860, 2002. doi:10.1006/eujc.2002.0586.