Practice 11

A **cut** of a graph is a partition of its vertices into exactly two disjoint sets. An edge **crosses** a cut if it starts in one set and ends in the other. Prove by contradiction that any cycle crosses a cut an even number of times.

Proof.

We proceed by contradiction.

Assume that some cycle crosses a cut an odd number of times. Pick a vertex a in the cycle; let S_0 be the set that contains a and S_1 be the other set of that cut.

After each cross of the cut, the set changes; thus when returning to a the set has changed an odd number of times. But crossing twice leaves the set unchanged, meaning $a \in S_1$. This contradicts our assumption that $a \in S_0$.

Because assuming that some cycle crossed a cut an odd number of times led to a contradiction, there must not be any such cycle. Hence, every cycle crosses a cut an even number of times.

□

PROBLEM 1 *Additional practice*

You should be able to prove all of the following

Theorem 1 *Removing a vertex (and all its associated edges) from a DAG leaves it a DAG.*

Theorem 2 *Removing an edge from a DAG leaves it a DAG.*

Theorem 3 *"can walk to" is a transitive relation on any graph.*

Theorem 4 *"can walk to in 1+ steps" is a transitive relation on any graph.*

Theorem 5 *"can walk to" is a reflexive relation on any graph.*

Theorem 6 *"can walk to in 1+ steps" is an irreflexive relation on any DAG.*

Theorem 7 *If "can walk to in 1+ steps" is irreflexive, then the graph is a DAG.*

Theorem 8 *"can walk to in 1+ steps" is an asymmetric relation on any DAG.*

Theorem 9 *If "can walk to in 1+ steps" is asymmetric, then the graph is a DAG.*

See also MCS problems 9.1, 9.3, 9.4, 9.6(c), 9.7(b), 9.8, 9.9, 9.11, 9.18, 9.19, 9.33, 9.36, 9.37 (but ignore linearity), 9.38, 9.39, 9.41, 9.42, 9.49, and 9.50.