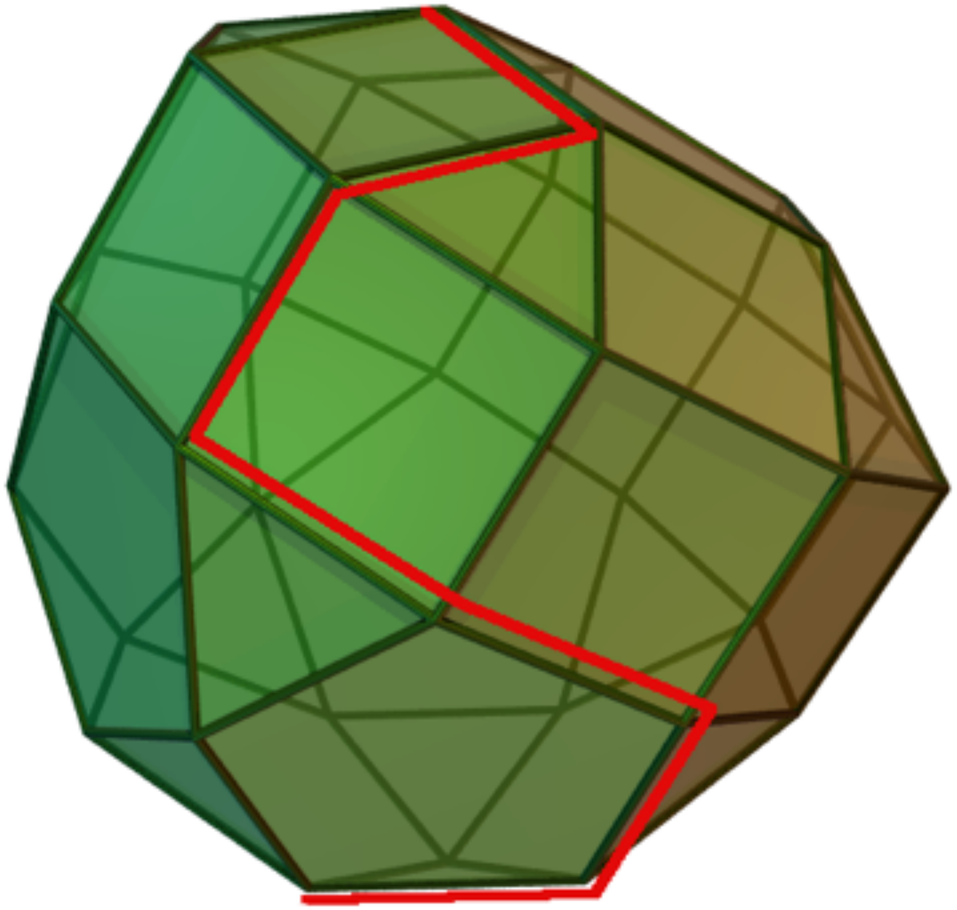
Simplex

CSC 282



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let v be any vertex of the feasible region
while there is a neighbor v' of v with better objective value:
    set v = v'
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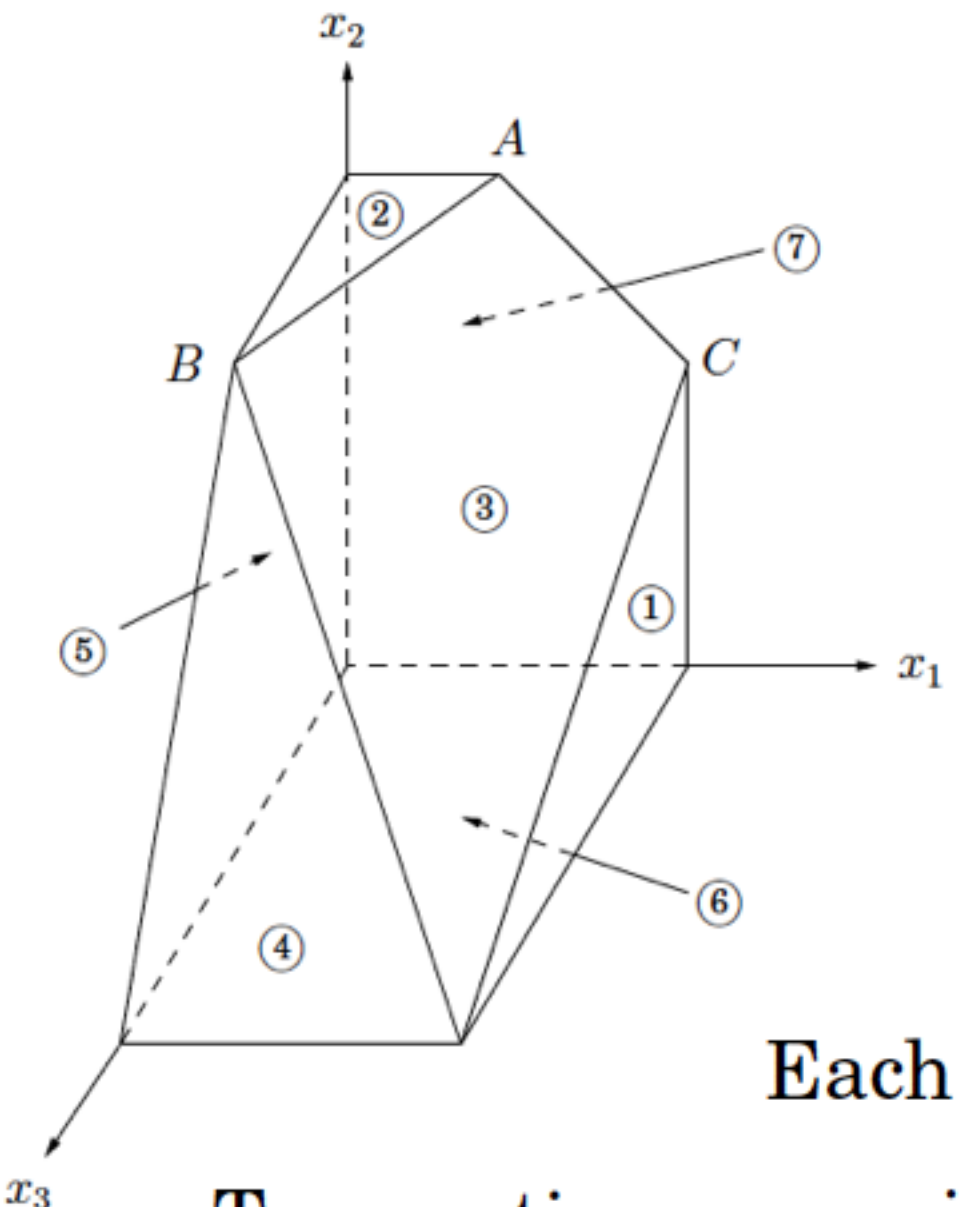


$$\max \quad x_1 + 6x_2 + 13x_3$$

$$x_1 \le 200 \quad ①$$

$$x_2 \le 300 \quad ②$$

$$x_1 + x_2 + x_3 \le 400 \quad ③$$

$$x_2 + 3x_3 \le 600 \quad ④$$

$$x_1 \ge 0 \quad ⑤$$

$$x_2 \ge 0 \quad ⑥$$

$$x_3 \ge 0 \quad ⑦$$

Each vertex is specified by a set of n inequalities.

Two vertices are *neighbors* if they have $n - 1$ defining inequalities in common.

Defining inequalities for A and C?

Case 1: Vertex is Origin

- Origin is optimal iff all $c_i \leq 0$
- Otherwise:
  - Release some tight constraint x_i
  - Increase x_i until some other inequality becomes tight

$$\max\ 2x_1 + 5x_2$$

$$\begin{aligned} 2x_1 - x_2 &\leq 4 & (1) \\ x_1 + 2x_2 &\leq 9 & (2) \\ -x_1 + x_2 &\leq 3 & (3) \\ x_1 &\geq 0 & (4) \\ x_2 &\geq 0 & (5) \end{aligned}$$

Increase x_2 until it "runs into" constraint 3 stopping at x_2=3

Case 2: Vertex is not the origin

- If not at the origin: transform coordinates so that the vertex is the origin

- New coordinate system **y** is a linear transformation of **x**

- New objective function becomes
  max $c_u + k^T y$

  - c_u is the value of the objective function at original vertex u

  - k is the transformed cost vector

Initial LP:

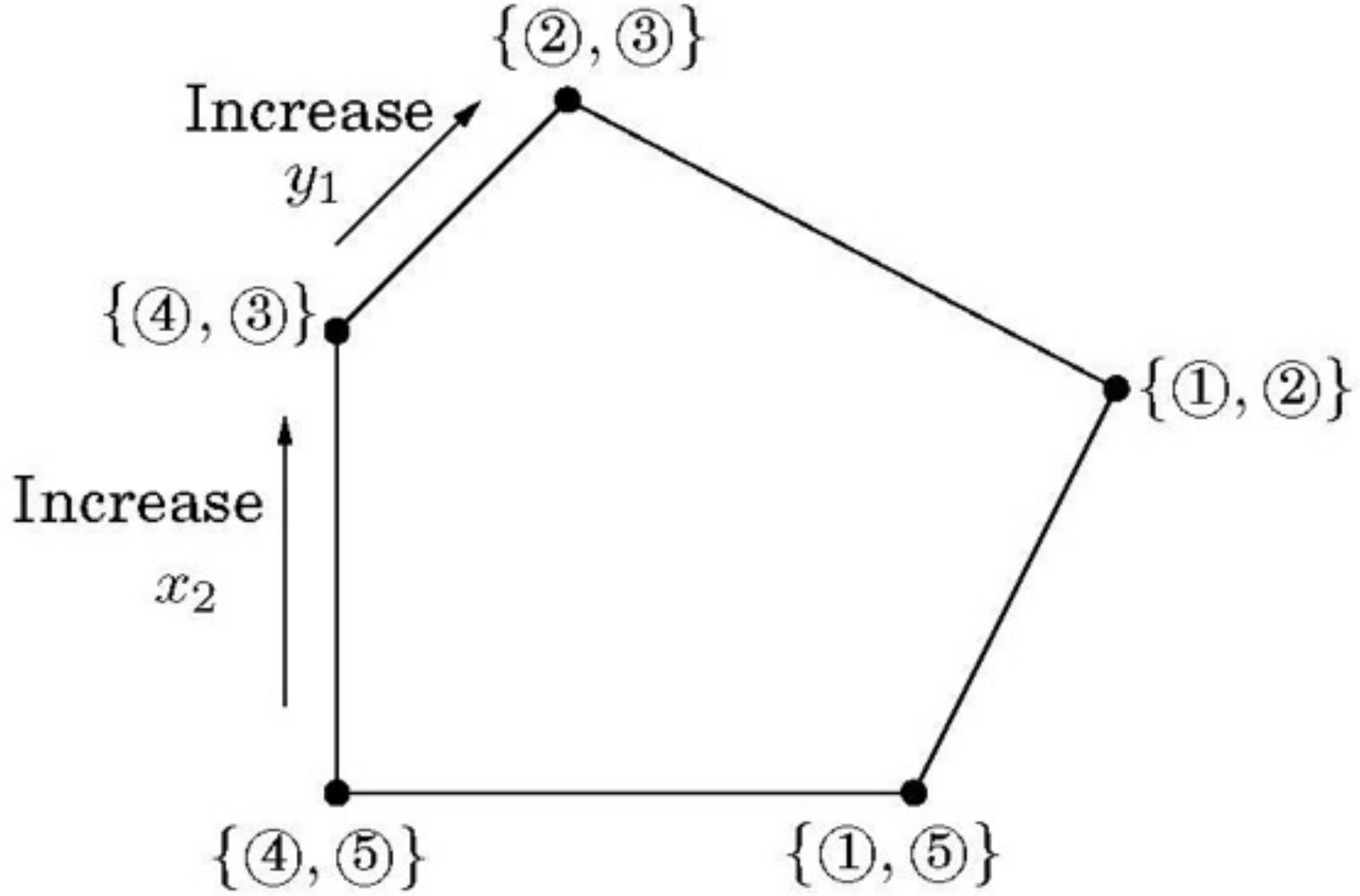
$$\max\ 2x_1 + 5x_2$$
$$\begin{aligned} 2x_1 - x_2 &\le 4 && ① \\ x_1 + 2x_2 &\le 9 && ② \\ -x_1 + x_2 &\le 3 && ③ \\ x_1 &\ge 0 && ④ \\ x_2 &\ge 0 && ⑤ \end{aligned}$$

Current vertex: {④, ⑤} (origin).
Objective value: 0.

Move: increase x_2.
⑤ is released, ③ becomes tight. Stop at $x_2 = 3$.

New vertex {④, ③} has local coordinates (y_1, y_2):

$$y_1 = x_1, \quad y_2 = 3 + x_1 - x_2$$



Rewritten LP:

$$\max\ 15 + 7y_1 - 5y_2$$

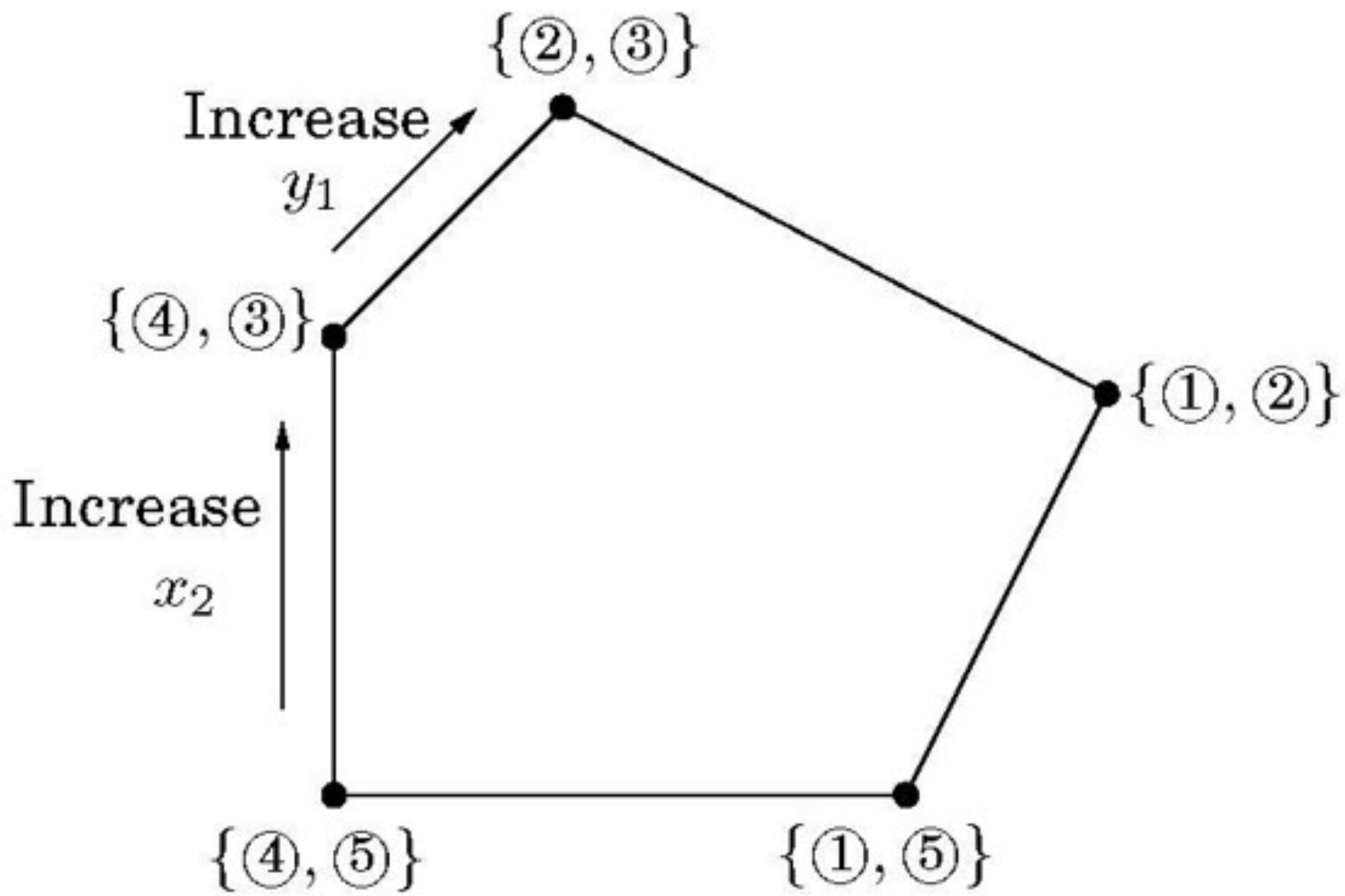
$$\begin{aligned} y_1 + y_2 &\le 7 && \text{①} \\ 3y_1 - 2y_2 &\le 3 && \text{②} \\ y_2 &\ge 0 && \text{③} \\ y_1 &\ge 0 && \text{④} \\ -y_1 + y_2 &\le 3 && \text{⑤} \end{aligned}$$

Current vertex: {④, ③}.
Objective value: 15.

Move: increase y_1.
④ is released, ② becomes tight. Stop at $y_1 = 1$.

New vertex {②, ③} has local coordinates (z_1, z_2):

$$z_1 = 3 - 3y_1 + 2y_2, \quad z_2 = y_2$$



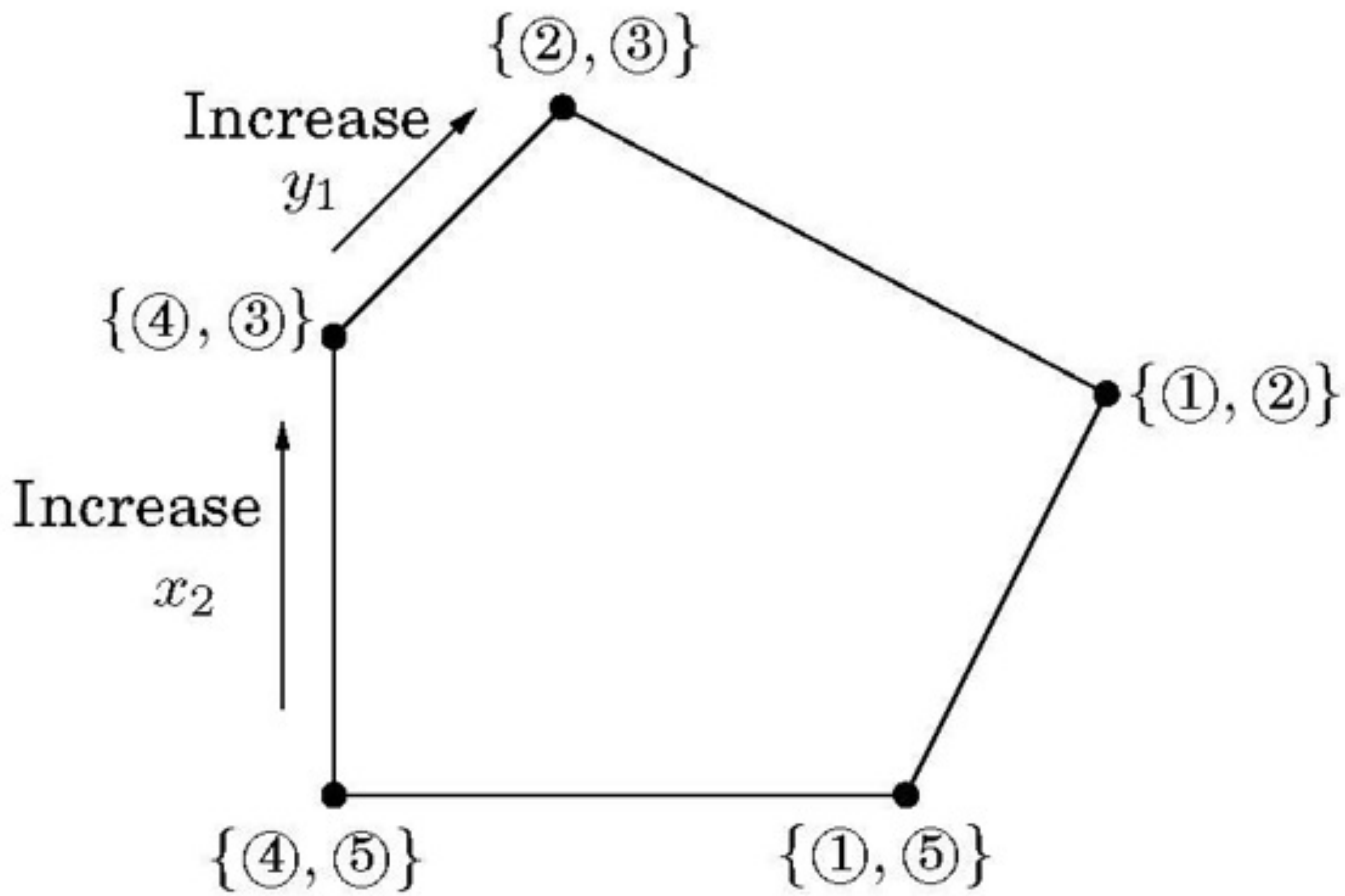
Rewritten LP:

$$
\begin{array}{rcll}
\max\ 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2 & & & \\
-\frac{1}{3}z_1 + \frac{5}{3}z_2 & \leq & 6 & \text{①} \\
z_1 & \geq & 0 & \text{②} \\
z_2 & \geq & 0 & \text{③} \\
\frac{1}{3}z_1 - \frac{2}{3}z_2 & \leq & 1 & \text{④} \\
\frac{1}{3}z_1 + \frac{1}{3}z_2 & \leq & 4 & \text{⑤}
\end{array}
$$

Current vertex: {②, ③}.
Objective value: 22.

Optimal: all $c_i < 0$.

Solve ②, ③ (in original LP) to get optimal solution $(x_1, x_2) = (1, 4)$.

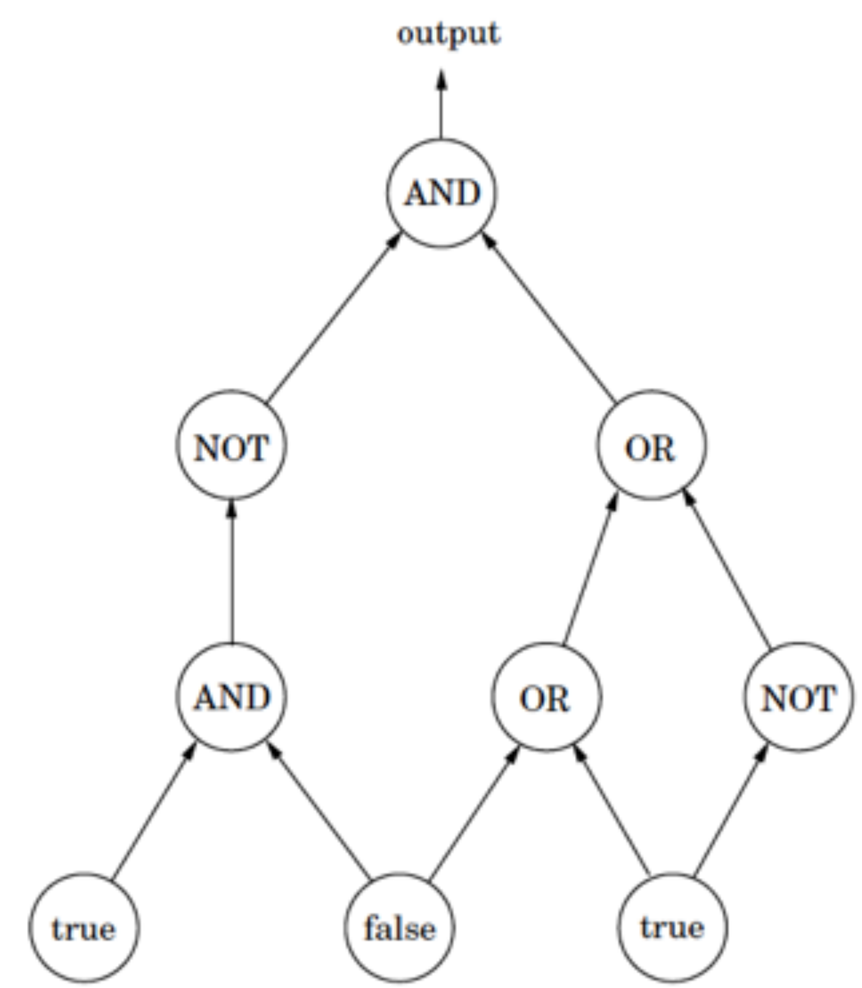


Running Time of Simplex

- n variables, m constraints
- Each iteration is O(mn)
  - Calculating objective value: O(n)
  - Checking if a neighbor is feasible:
    - Naive approach $O(mn^4)$
    - Incremental algorithm amortized cost O(mn)
  - Moving to a neighbor: O(1)
- Worst case number of iterations $\binom{m+n}{n}$ **exponential**

Circuit Evaluation

- Given Boolean circuit and its inputs, compute the output
- Can be encoded as an LP
- Shows that LP is "P-complete" - as hard as any program in P



Circuit Satisfiability

- Given Boolean circuit, is there some set of inputs that makes the output 1?
- Cannot be encoded as an LP
- Can be encoded as an integer program
- Shows that integer programming is “NP-complete” - as hard as any program in NP