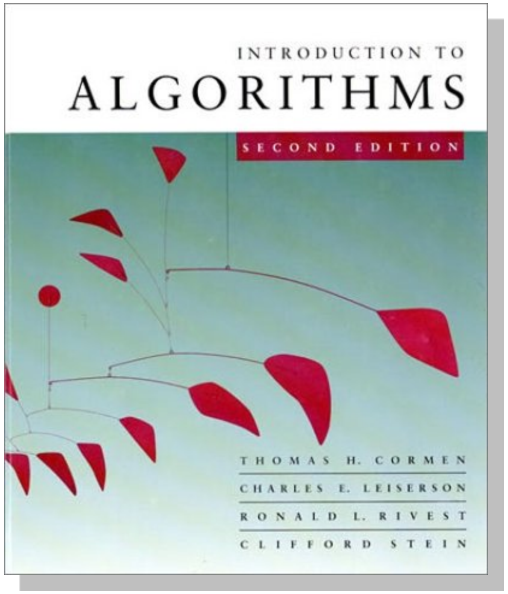
Introduction to Algorithms
6.046J/18.401J



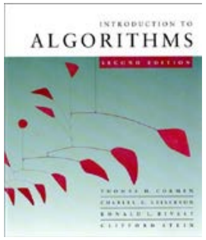
LECTURE 14
Competitive Analysis
- Self-organizing lists
- Move-to-front heuristic
- Competitive analysis of MTF

Prof. Charles E. Leiserson

Self-organizing lists

List L of n elements

- The operation ACCESS(x) costs $\text{rank}_L(x) =$ distance of x from the head of L.
- L can be reordered by transposing adjacent elements at a cost of 1.



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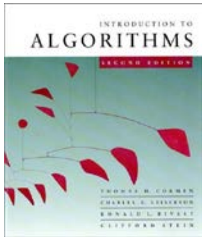
Example:

$$L \rightarrow \boxed{12} \rightarrow \boxed{3} \rightarrow \boxed{50} \rightarrow \boxed{14} \rightarrow \boxed{17} \rightarrow \boxed{4}$$



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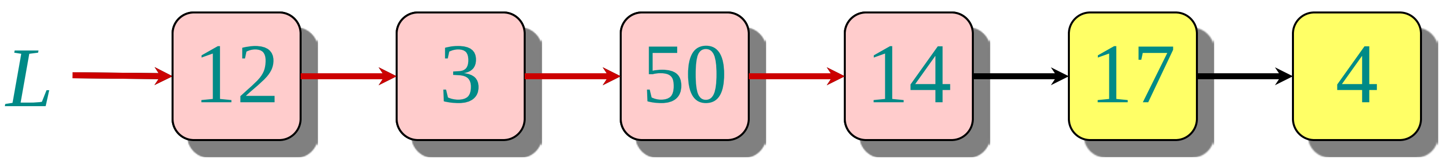
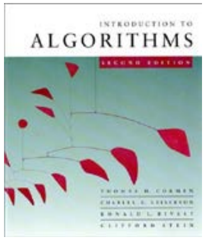


Accessing the element with key 14 costs 4.



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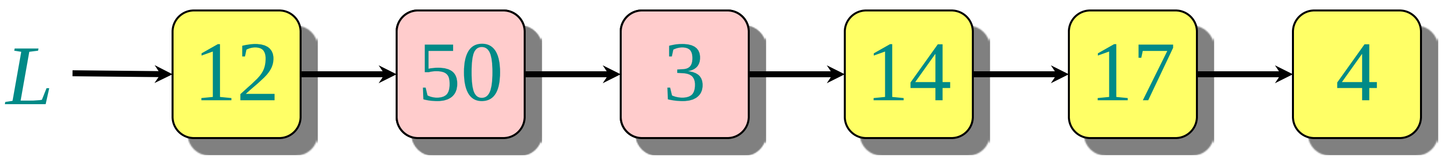
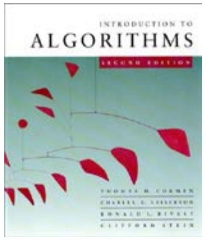


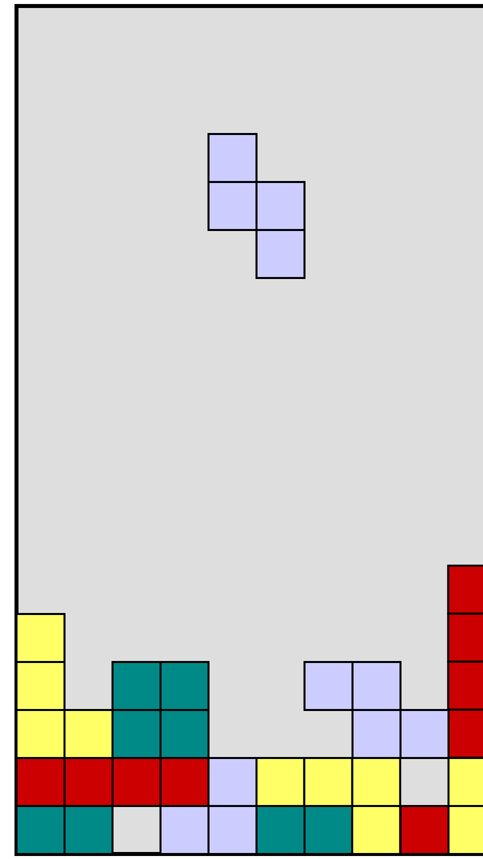
Transposing 3 and 50 costs 1.



On-line and off-line problems

Definition. A sequence S of operations is provided one at a time. For each operation, an ***on-line*** algorithm A must execute the operation immediately without any knowledge of future operations (e.g., *Tetris*).

An ***off-line*** algorithm may see the whole sequence S in advance.



The game of Tetris

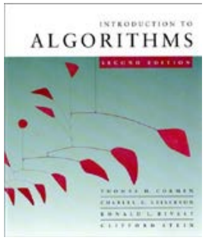
Goal: Minimize the total cost $C_A(S)$.

Worst-case analysis of self-organizing lists

An adversary always accesses the tail (nth) element of L. Then, for any on-line algorithm A, we have

$$C_A(S) = \Omega(|S| \cdot n)$$

in the worst case.



Average-case analysis of self-organizing lists

Suppose that element x is accessed with probability $p(x)$. Then, we have

$$\mathrm{E}[C_A(S)] = \sum_{x \in L} p(x) \cdot \mathrm{rank}_L(x),$$

which is minimized when L is sorted in decreasing order with respect to p.

Heuristic: Keep a count of the number of times each element is accessed, and maintain L in order of decreasing count.

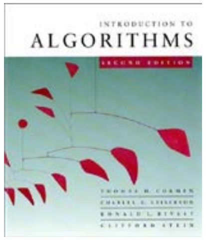
The move-to-front heuristic

Practice: Implementers discovered that the ***move-to-front (MTF)*** heuristic empirically yields good results.

IDEA: After accessing x, move x to the head of L using transposes:

$$\text{cost} = 2 \cdot \text{rank}_L(x) \; .$$

The MTF heuristic responds well to locality in the access sequence S.



Competitive analysis

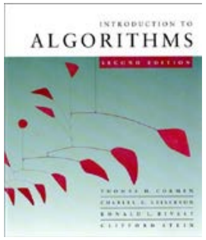
Definition. An on-line algorithm A is *α-competitive* if there exists a constant k such that for any sequence S of operations,

$$C_A(S) \le \alpha \cdot C_{\mathrm{OPT}}(S) + k\,,$$

where OPT is the optimal off-line algorithm ("God's algorithm").

MTF is O(1)-competitive

Theorem. MTF is 4-competitive for self-organizing lists.



MTF is O(1)-competitive

Theorem. MTF is 4-competitive for self-organizing lists.

Proof. Let L_i be MTF's list after the ith access, and let $L_i{*}$ be OPT's list after the ith access.

$$
\begin{aligned}
\text{Let } c_i &= \text{MTF's cost for the } i\text{th operation} \\
&= 2 \cdot \text{rank}_{L_{i-1}}(x) \text{ if it accesses } x; \\
c_i{*} &= \text{OPT's cost for the } i\text{th operation} \\
&= \text{rank}_{L_{i-1}{*}}(x) + t_i \,,
\end{aligned}
$$

where t_i is the number of transposes that OPT performs.

Potential function

Define the potential function $\Phi:\{L_i\} \to \mathsf{R}$ by

$$\Phi(L_i) = 2 \cdot |\{(x, y) : x \prec_{L_i} y \text{ and } y \prec_{L_i{}^*} x\}|$$
$$= 2 \cdot \#\ \textbf{\textit{inversions}}\ .$$

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Example.

$L_i \rightarrow \boxed{E} \rightarrow \boxed{C} \rightarrow \boxed{A} \rightarrow \boxed{D} \rightarrow \boxed{B}$

$L_i^* \rightarrow \boxed{C} \rightarrow \boxed{A} \rightarrow \boxed{B} \rightarrow \boxed{D} \rightarrow \boxed{E}$

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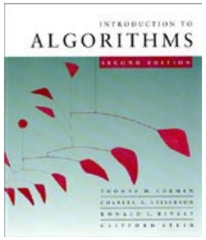
$$\Phi(L_i) = 2 \cdot |\{\ldots\}|$$



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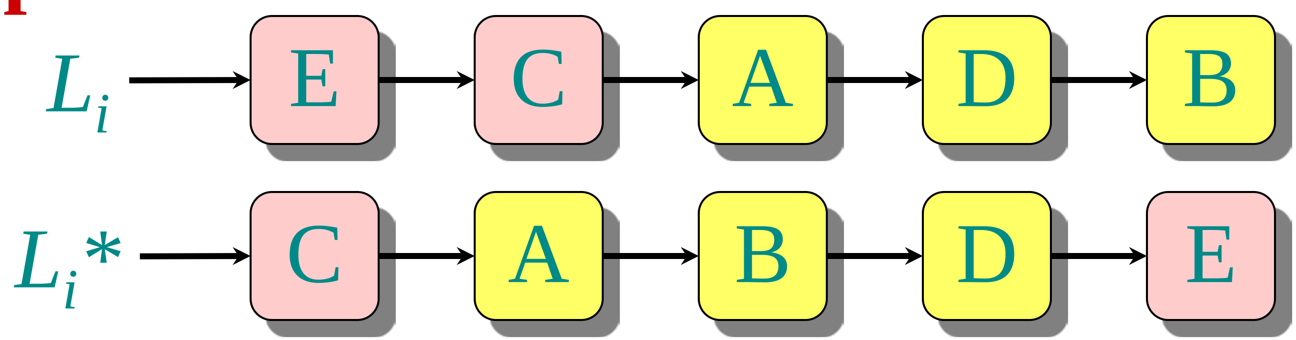

$$\Phi(L_i) = 2 \cdot |\{(\mathrm{E,C}), \ldots\}|$$

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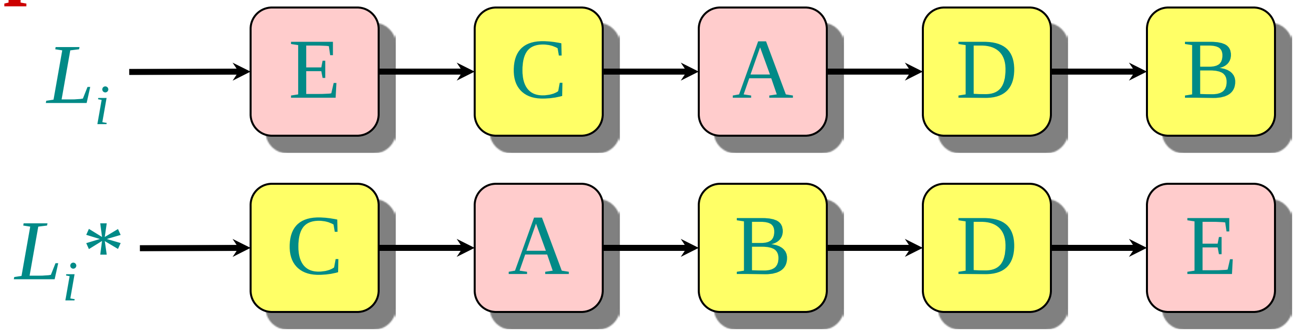
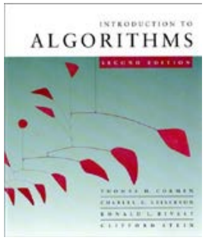


$$\Phi(L_i) = 2 \cdot |\{(\mathrm{E},\mathrm{C}), (\mathrm{E},\mathrm{A}), \ldots\}|$$



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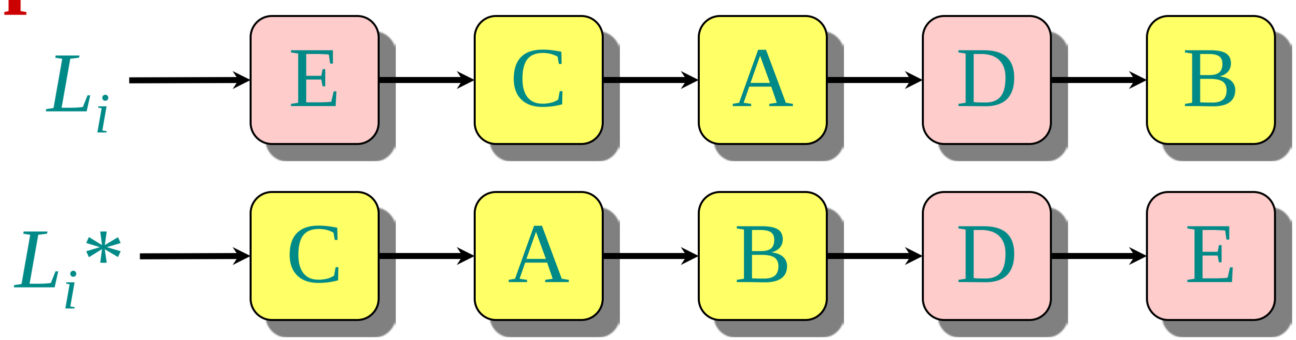

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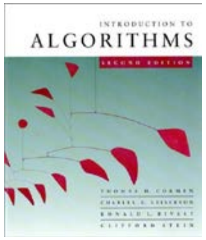

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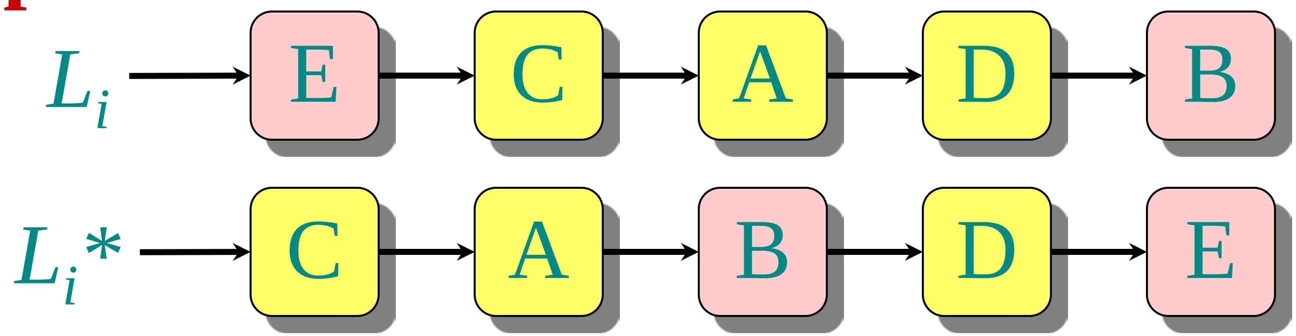

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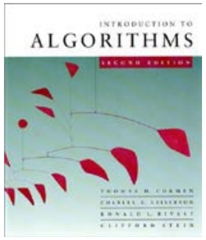
$$= 10 \,.$$



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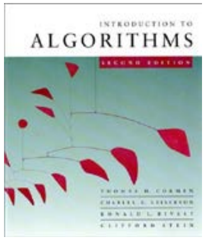

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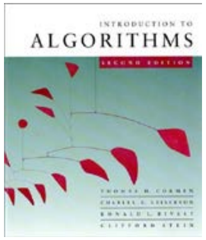
Note that

- $\Phi(L_i) \geq 0$ for $i = 0, 1, \ldots,$
- $\Phi(L_0) = 0$ if MTF and OPT start with the same list.



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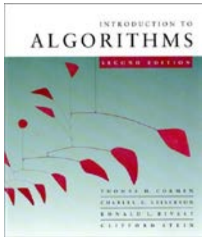
How much does Φ change from 1 transpose?

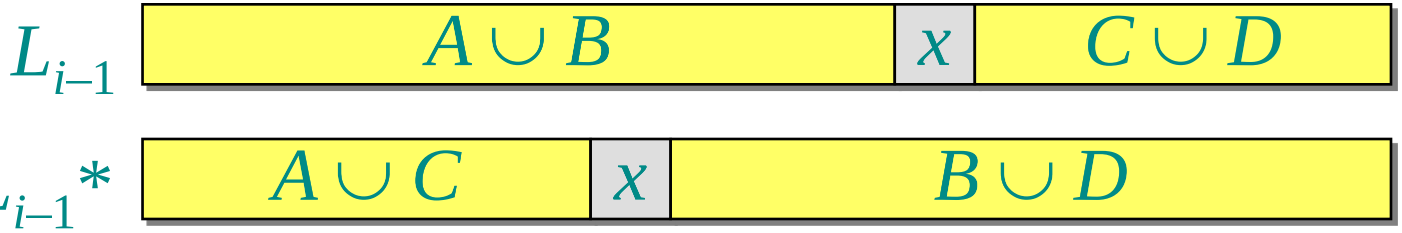
- A transpose creates/destroys 1 inversion.
- $\Delta\Phi = \pm 2$.

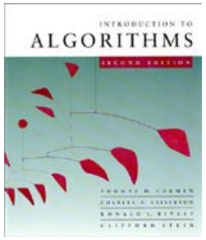


What happens on an access?

Suppose that operation i accesses element x, and define

$$A = \{y \in L_{i-1} : y \prec_{L_{i-1}} x \text{ and } y \prec_{L_{i-1}^*} x\},$$
$$B = \{y \in L_{i-1} : y \prec_{L_{i-1}} x \text{ and } y \succ_{L_{i-1}^*} x\},$$
$$C = \{y \in L_{i-1} : y \succ_{L_{i-1}} x \text{ and } y \prec_{L_{i-1}^*} x\},$$
$$D = \{y \in L_{i-1} : y \succ_{L_{i-1}} x \text{ and } y \succ_{L_{i-1}^*} x\}.$$





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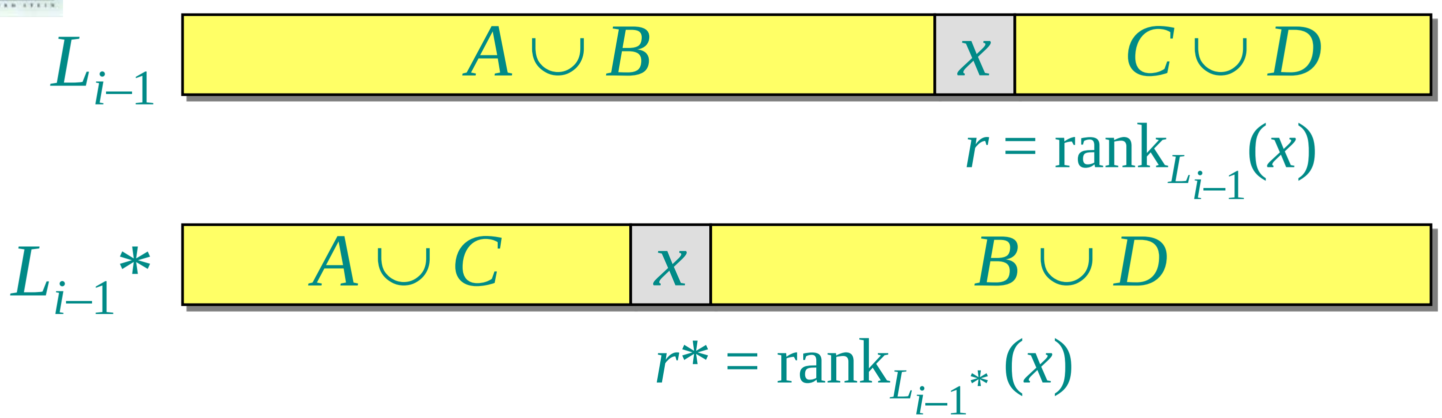
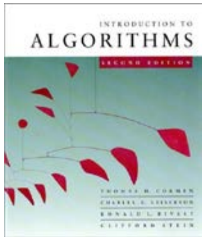


We have $r = |A| + |B| + 1$ and $r^* = |A| + |C| + 1$.



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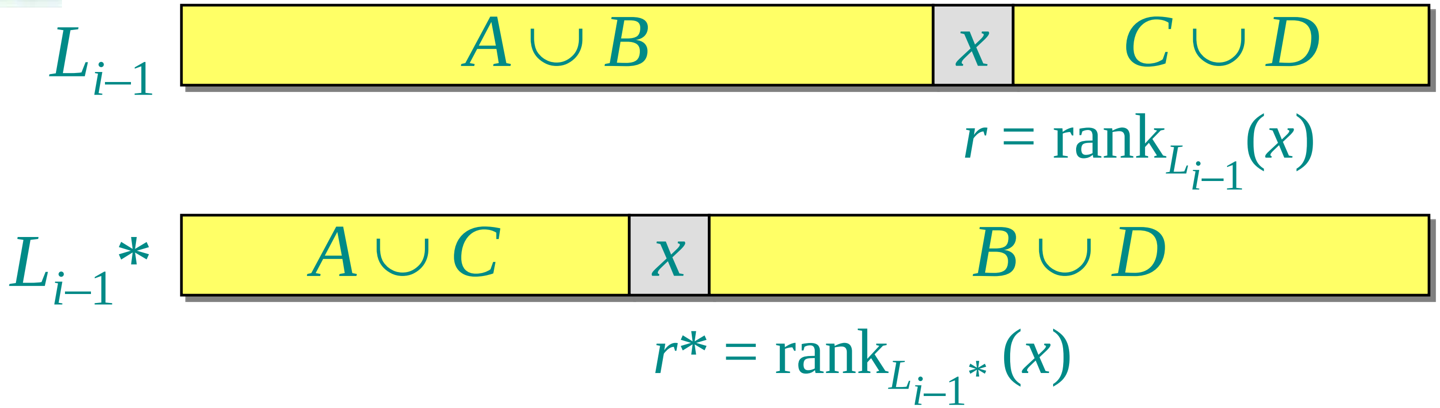

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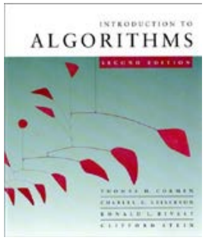
When MTF moves x to the front, it creates $|A|$ inversions and destroys $|B|$ inversions. Each transpose by OPT creates ≤ 1 inversion. Thus, we have

$$\Phi(L_i) - \Phi(L_{i-1}) \leq 2(|A| - |B| + t_i) \ .$$



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The amortized cost for the ith operation of MTF with respect to Φ is

$$\hat{c}_i = c_i + \Phi(L_i) - \Phi(L_{i-1})$$

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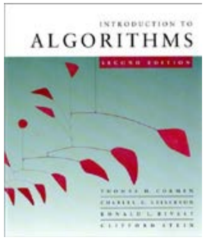

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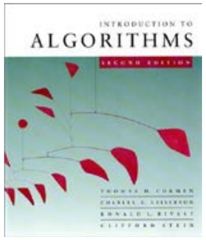
(since $r = |A| + |B| + 1$)



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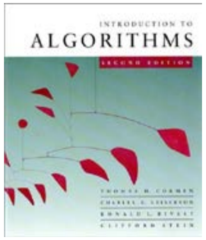

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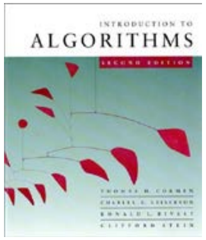

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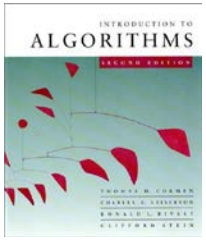
&\le 4(r^* + t_i)
\end{aligned}$$

(since $r^* = |A| + |C| + 1 \ge |A| + 1$)



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&\le 4(r^* + t_i) \\
&= 4c_i^*.
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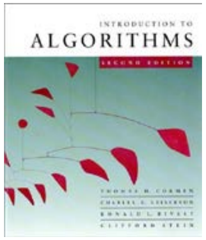
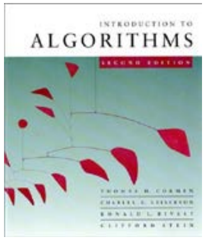


The grand finale

Thus, we have

$$C_{\mathrm{MTF}}(S) = \sum_{i=1}^{|S|} c_i$$



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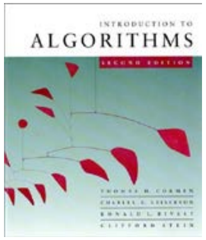

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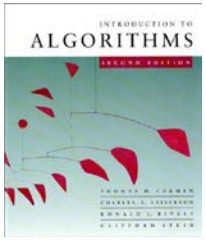
&\le \left(\sum_{i=1}^{|S|} 4c_i^* \right) + \Phi(L_0) - \Phi(L_{|S|})
\end{aligned}
$$



The grand finale

Thus, we have

$$
\begin{aligned}
C_{\mathrm{MTF}}(S) &= \sum_{i=1}^{|S|} c_i \\
&= \sum_{i=1}^{|S|} \left(\hat{c}_i + \Phi(L_{i-1}) - \Phi(L_i) \right) \\
&\le \left(\sum_{i=1}^{|S|} 4 c_i^* \right) + \Phi(L_0) - \Phi(L_{|S|}) \\
&\le 4 \cdot C_{\mathrm{OPT}}(S),
\end{aligned}
$$

since $\Phi(L_0) = 0$ and $\Phi(L_{|S|}) \ge 0$. $\square$

Addendum

If we count transpositions that move x toward the front as "free" (models splicing x in and out of L in constant time), then MTF is 2-competitive.

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What if $L_0 \neq L_0{}^*$?

- Then, $\Phi(L_0)$ might be $\Theta(n^2)$ in the worst case.
- Thus, $C_{\mathrm{MTF}}(S) \leq 4 \cdot C_{\mathrm{OPT}}(S) + \Theta(n^2)$, which is still 4-competitive, since n^2 is constant as $|S| \to \infty$.