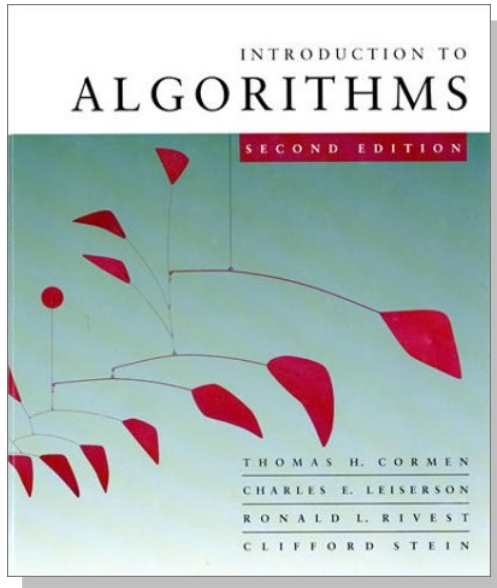


Introduction to Algorithms

6.046J/18.401J

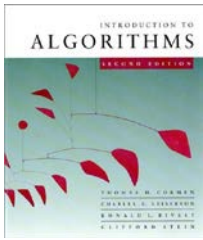


LECTURE 15

Dynamic Programming

- Longest common subsequence
- Optimal substructure
- Overlapping subproblems

Prof. Charles E. Leiserson

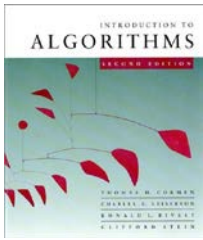


Dynamic programming

Design technique, like divide-and-conquer.

Example: Longest Common Subsequence (LCS)

- Given two sequences $x[1 \dots m]$ and $y[1 \dots n]$, find a longest subsequence common to them both.



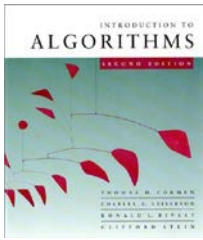
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x : A B C B D A B

y : B D C A B A



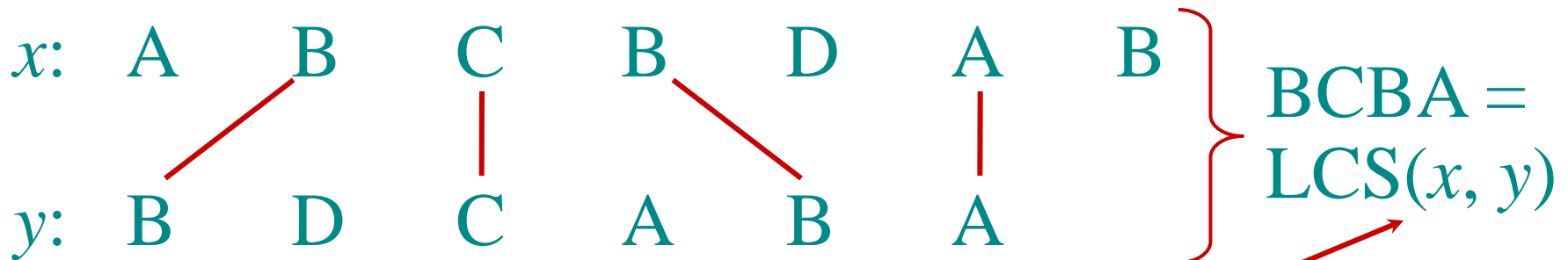
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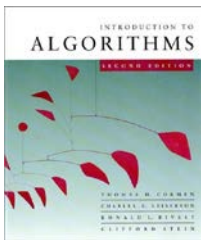


functional notation,
but not a function



Brute-force LCS algorithm

Check every subsequence of $x[1 \dots m]$ to see if it is also a subsequence of $y[1 \dots n]$.



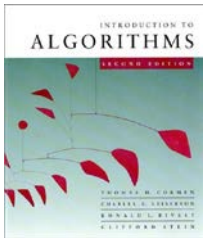
Brute-force LCS algorithm

Check every subsequence of $x[1 \dots m]$ to see if it is also a subsequence of $y[1 \dots n]$.

Analysis

- Checking = $O(n)$ time per subsequence.
- 2^m subsequences of x (each bit-vector of length m determines a distinct subsequence of x).

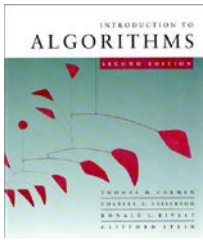
Worst-case running time = $O(n2^m)$
= exponential time.



Towards a better algorithm

Simplification:

1. Look at the *length* of a longest-common subsequence.
2. Extend the algorithm to find the LCS itself.

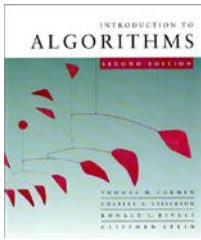


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Notation: Denote the length of a sequence s by $|s|$.



Towards a better algorithm

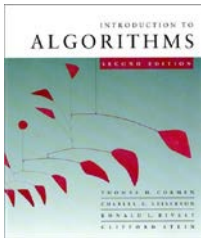
Simplification:

1. Look at the *length* of a longest-common subsequence.
2. Extend the algorithm to find the LCS itself.

Notation: Denote the length of a sequence s by $|s|$.

Strategy: Consider *prefixes* of x and y .

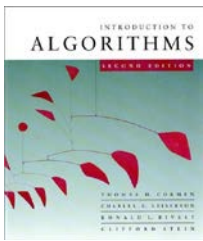
- Define $c[i, j] = |\text{LCS}(x[1 \dots i], y[1 \dots j])|$.
- Then, $c[m, n] = |\text{LCS}(x, y)|$.



Recursive formulation

Theorem.

$$c[i, j] = \begin{cases} c[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max\{c[i-1, j], c[i, j-1]\} & \text{otherwise.} \end{cases}$$

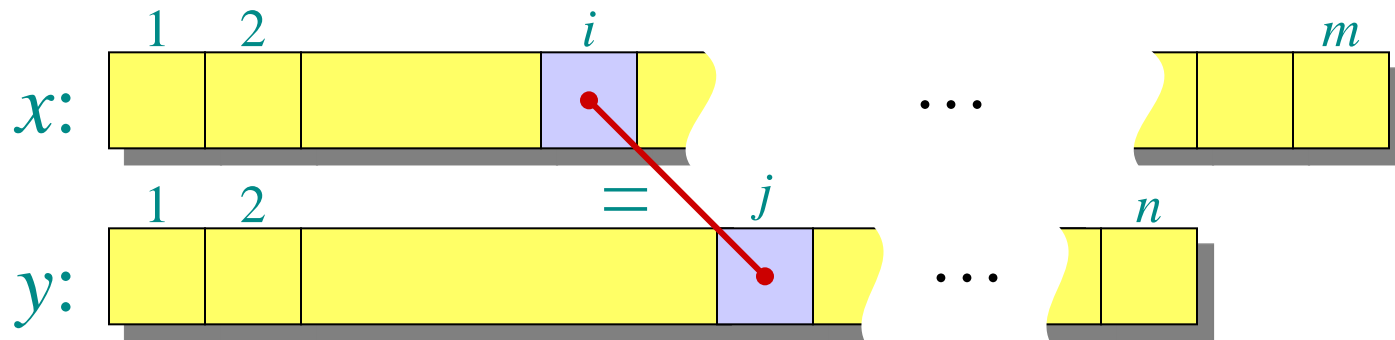


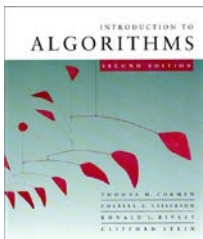
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Proof. Case $x[i] = y[j]$:



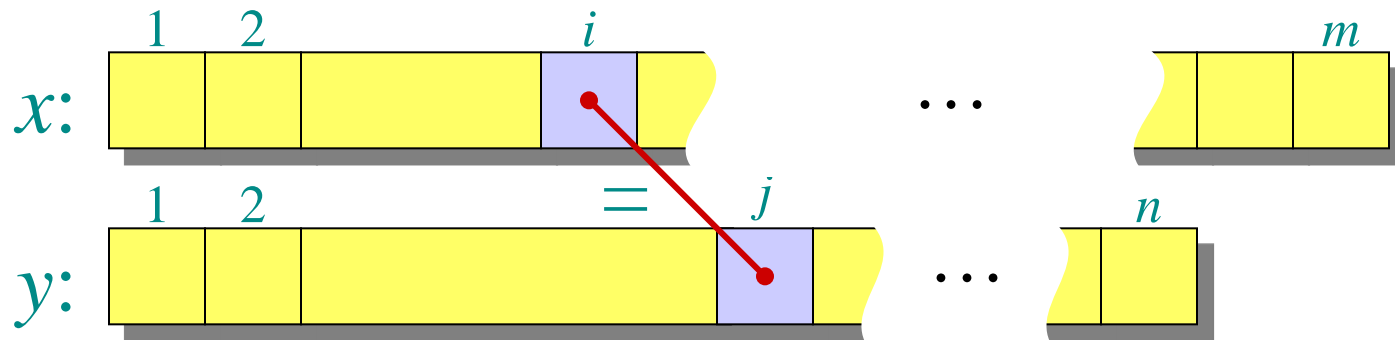


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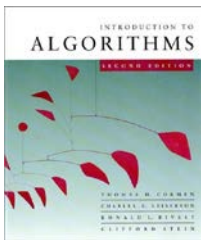
Let $z[1 \dots k] = \text{LCS}(x[1 \dots i], y[1 \dots j])$, where $c[i, j] = k$. Then, $z[k] = x[i]$, or else z could be extended. Thus, $z[1 \dots k-1]$ is CS of $x[1 \dots i-1]$ and $y[1 \dots j-1]$.



Proof (continued)

Claim: $z[1 \dots k-1] = \text{LCS}(x[1 \dots i-1], y[1 \dots j-1])$.

Suppose w is a longer CS of $x[1 \dots i-1]$ and $y[1 \dots j-1]$, that is, $|w| > k-1$. Then, *cut and paste*: $w \parallel z[k]$ (w concatenated with $z[k]$) is a common subsequence of $x[1 \dots i]$ and $y[1 \dots j]$ with $|w \parallel z[k]| > k$. Contradiction, proving the claim.



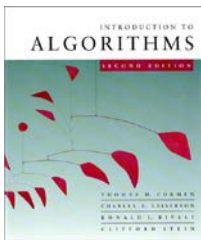
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Thus, $c[i-1, j-1] = k-1$, which implies that $c[i, j] = c[i-1, j-1] + 1$.

Other cases are similar. □



Dynamic-programming hallmark #1

Optimal substructure

*An optimal solution to a problem
(instance) contains optimal
solutions to subproblems.*

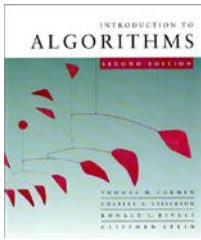


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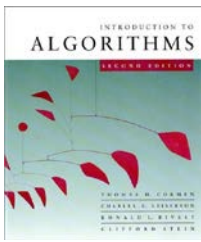
*An optimal solution to a problem
(instance) contains optimal
solutions to subproblems.*

If $z = \text{LCS}(x, y)$, then any prefix of z is
an LCS of a prefix of x and a prefix of y .



Recursive algorithm for LCS

```
LCS( $x, y, i, j$ ) // ignoring base cases
  if  $x[i] = y[j]$ 
    then  $c[i, j] \leftarrow \text{LCS}(x, y, i-1, j-1) + 1$ 
    else  $c[i, j] \leftarrow \max \{ \text{LCS}(x, y, i-1, j),$ 
                                    $\text{LCS}(x, y, i, j-1) \}$ 
  return  $c[i, j]$ 
```



Recursive algorithm for LCS

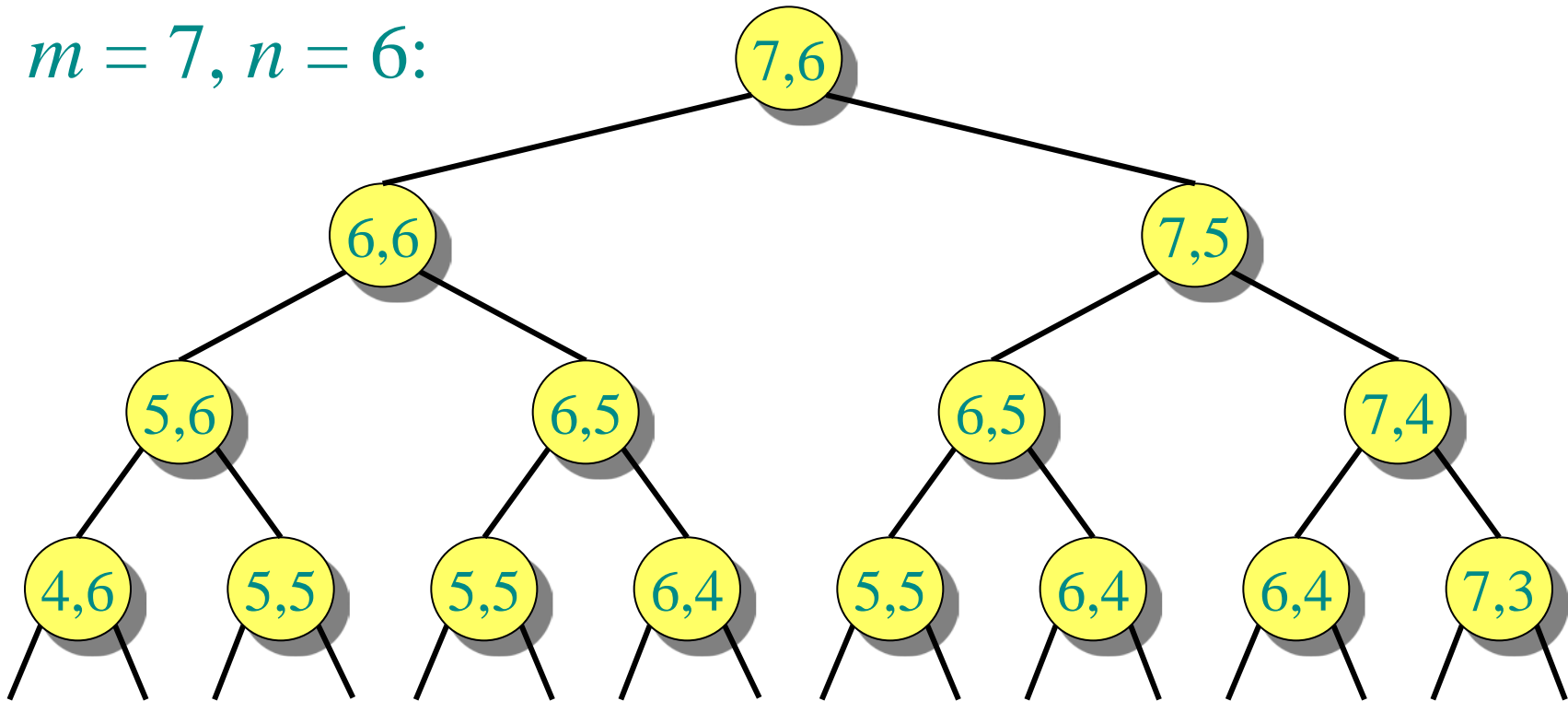
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  return  $c[i, j]$ 
```

Worse case: $x[i] \neq y[j]$, in which case the algorithm evaluates two subproblems, each with only one parameter decremented.



Recursion tree

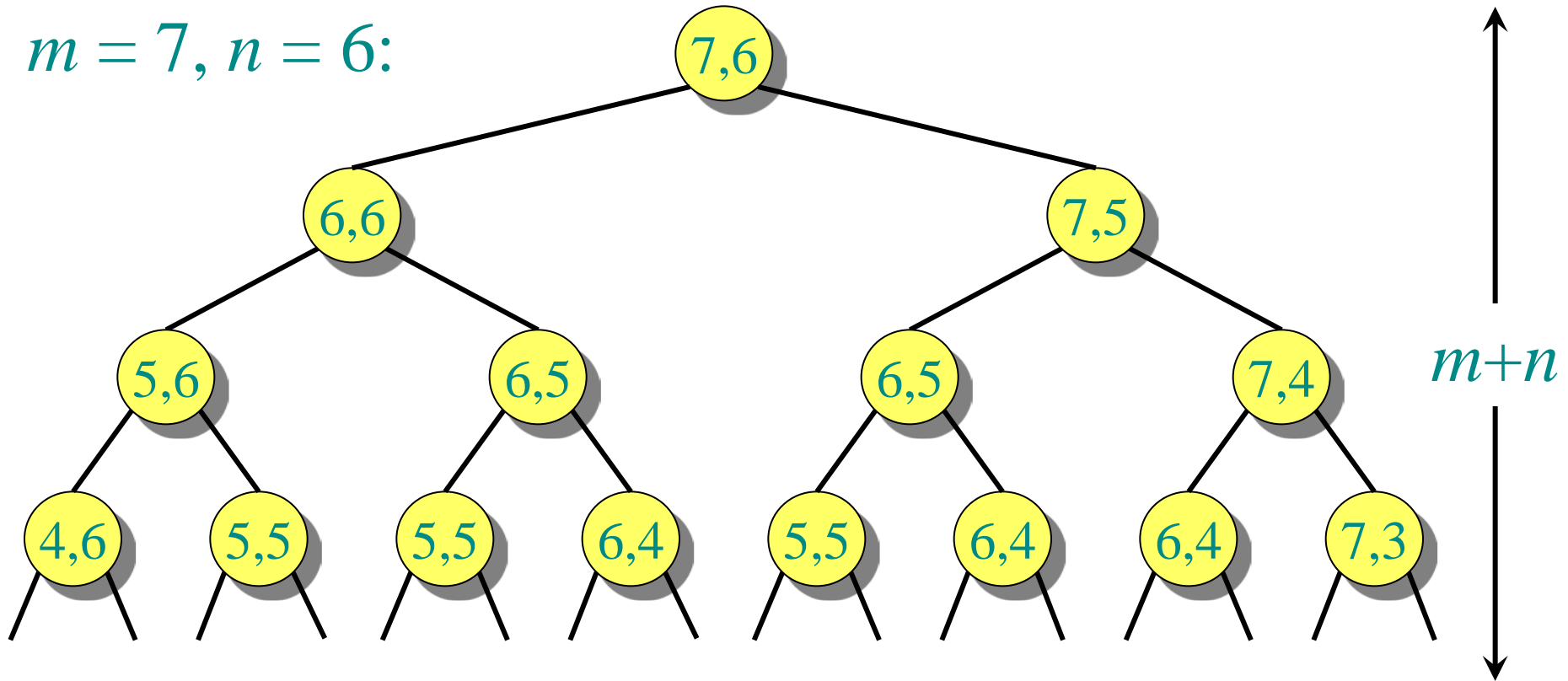
$m = 7, n = 6$:





Recursion tree

$m = 7, n = 6:$

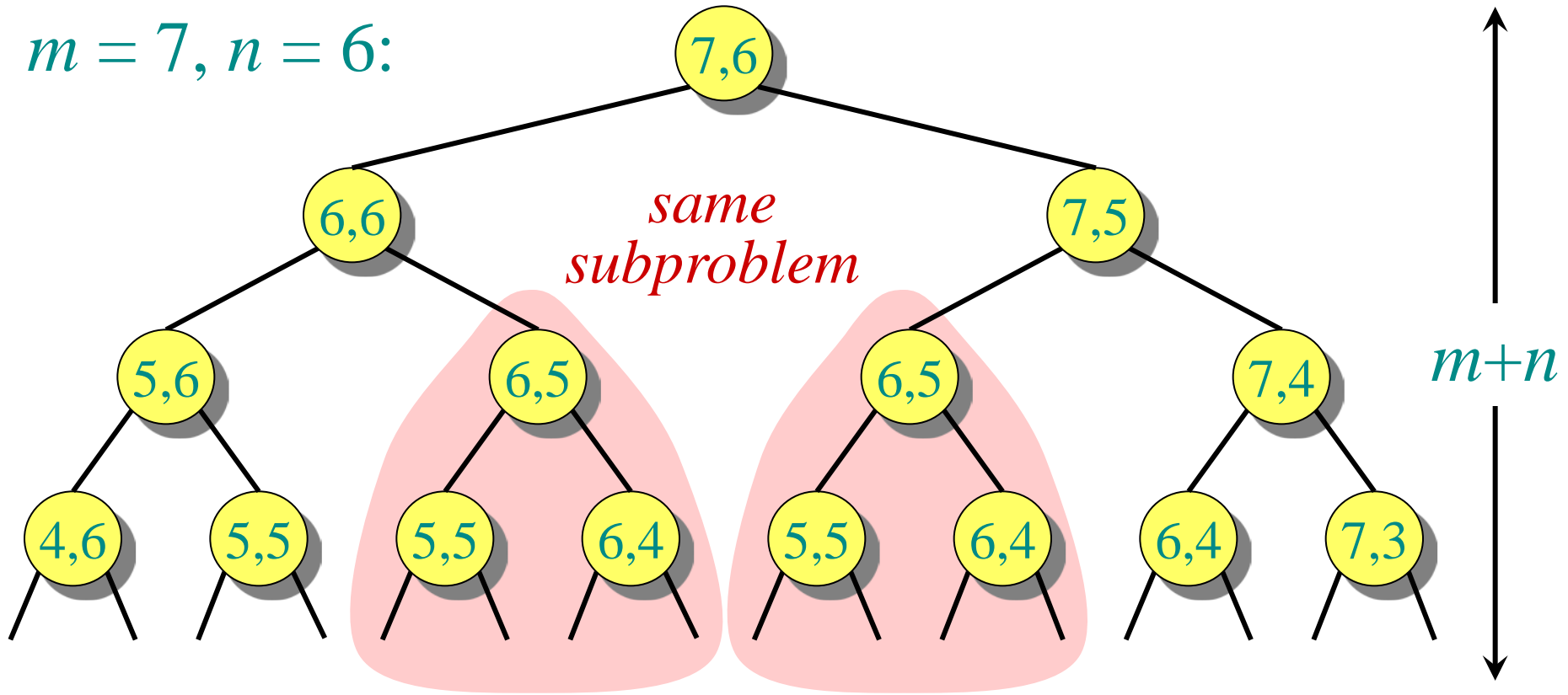


Height = $m + n \Rightarrow$ work potentially exponential.

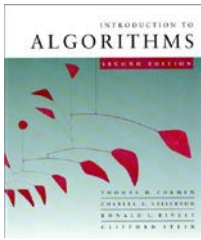


Recursion tree

$m = 7, n = 6:$



Height = $m + n \Rightarrow$ work potentially exponential,
but we're solving subproblems already solved!



Dynamic-programming hallmark #2

Overlapping subproblems

A recursive solution contains a “small” number of distinct subproblems repeated many times.

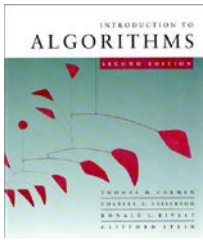


Dynamic-programming hallmark #2

Overlapping subproblems

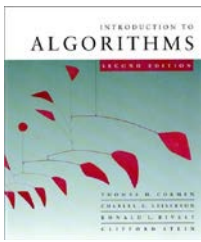
A recursive solution contains a “small” number of distinct subproblems repeated many times.

The number of distinct LCS subproblems for two strings of lengths m and n is only mn .



Memoization algorithm

Memoization: After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.



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$\text{LCS}(x, y, i, j)$

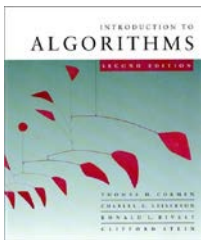
if $c[i, j] = \text{NIL}$

then if $x[i] = y[j]$

then $c[i, j] \leftarrow \text{LCS}(x, y, i-1, j-1) + 1$

else $c[i, j] \leftarrow \max \{ \text{LCS}(x, y, i-1, j), \text{LCS}(x, y, i, j-1) \}$

*same
as
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*same
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Time = $\Theta(mn)$ = constant work per table entry.

Space = $\Theta(mn)$.



Dynamic-programming algorithm

IDEA:

Compute the table bottom-up.

| | | A | B | C | B | D | A | B | |
|---|--|---|---|---|---|---|---|---|---|
| | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| B | | 0 | 0 | 1 | 1 | 1 | 1 | 1 | |
| D | | 0 | 0 | 1 | 1 | 1 | 2 | 2 | |
| C | | 0 | 0 | 1 | 2 | 2 | 2 | 2 | |
| A | | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| B | | 0 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| A | | 0 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |



Dynamic-programming algorithm

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|---|---|---|---|---|---|---|---|---|
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| B | 0 | 0 | 1 | 1 | 1 | 1 | 1 | |
| D | 0 | 0 | 1 | 1 | 1 | 2 | 2 | |
| C | 0 | 0 | 1 | 2 | 2 | 2 | 2 | |
| A | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| B | 0 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| A | 0 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |



Dynamic-programming algorithm

IDEA:

Compute the table bottom-up.

Time = $\Theta(mn)$.

Reconstruct LCS by tracing backwards.

| | A | B | C | B | D | A | B |
|---|---|---|---|---|---|---|---|
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| D | 0 | 0 | 1 | 1 | 1 | 2 | 2 |
| C | 0 | 0 | 1 | 2 | 2 | 2 | 2 |
| A | 0 | 1 | 1 | 2 | 2 | 2 | 3 |
| B | 0 | 1 | 2 | 2 | 3 | 3 | 4 |
| A | 0 | 1 | 2 | 2 | 3 | 3 | 4 |



Dynamic-programming algorithm

IDEA:

Compute the table bottom-up.

Time = $\Theta(mn)$.

Reconstruct LCS by tracing backwards.

Space = $\Theta(mn)$.

Exercise:

$O(\min\{m, n\})$.

| | A | B | C | B | D | A | B |
|---|---|---|---|---|---|---|---|
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| D | 0 | 0 | 1 | 1 | 1 | 2 | 2 |
| C | 0 | 0 | 1 | 2 | 2 | 2 | 2 |
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