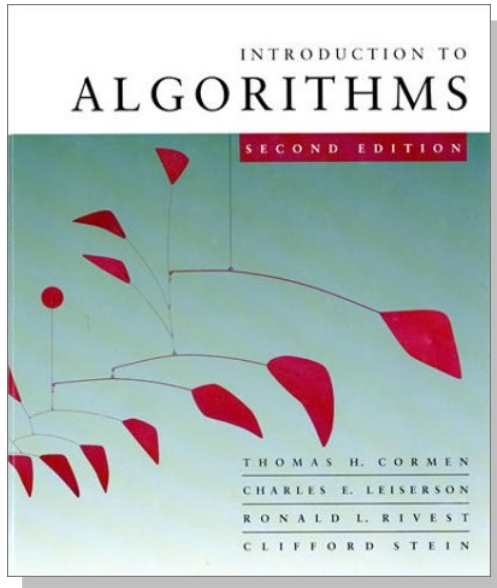


Introduction to Algorithms

6.046J/18.401J

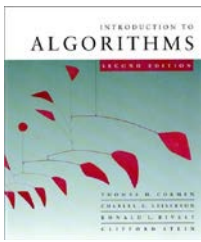


LECTURE 17

Shortest Paths I

- Properties of shortest paths
- Dijkstra's algorithm
- Correctness
- Analysis
- Breadth-first search

Prof. Erik Demaine



Paths in graphs

Consider a digraph $G = (V, E)$ with edge-weight function $w : E \rightarrow \mathbb{R}$. The *weight* of path $p = v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k$ is defined to be

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1}).$$

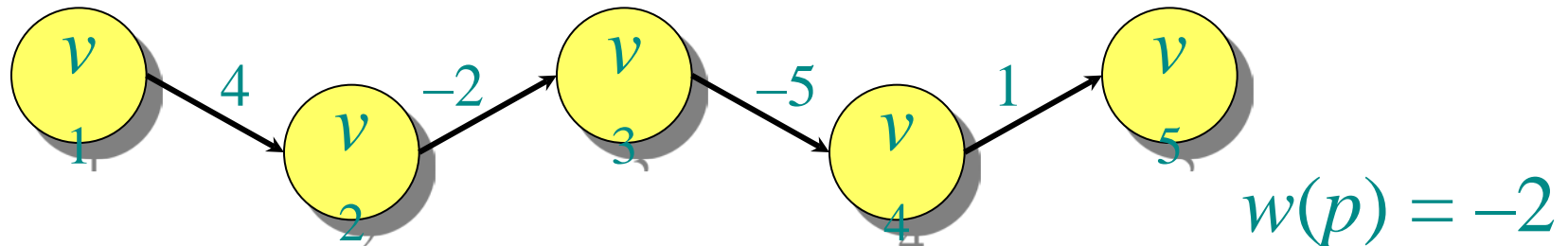


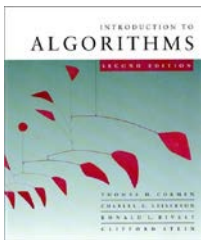
Paths in graphs

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Example:



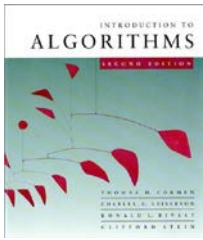


Shortest paths

A *shortest path* from u to v is a path of minimum weight from u to v . The *shortest-path weight* from u to v is defined as

$$\delta(u, v) = \min\{w(p) : p \text{ is a path from } u \text{ to } v\}.$$

Note: $\delta(u, v) = \infty$ if no path from u to v exists.



Well-definedness of shortest paths

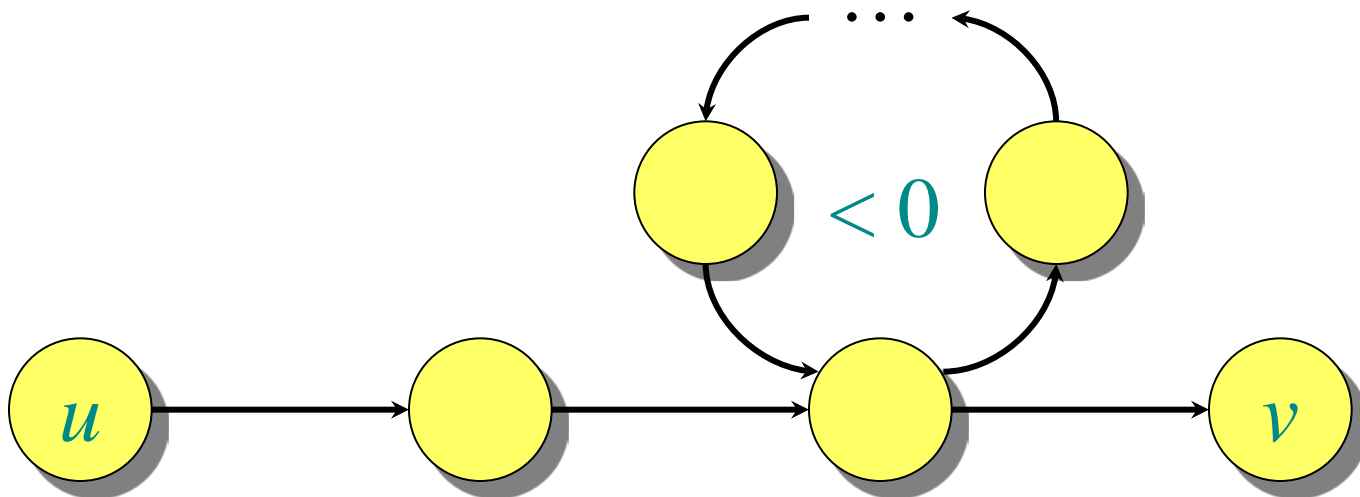
If a graph G contains a negative-weight cycle, then some shortest paths do not exist.

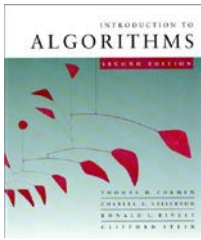


Well-definedness of shortest paths

If a graph G contains a negative-weight cycle, then some shortest paths do not exist.

Example:





Optimal substructure

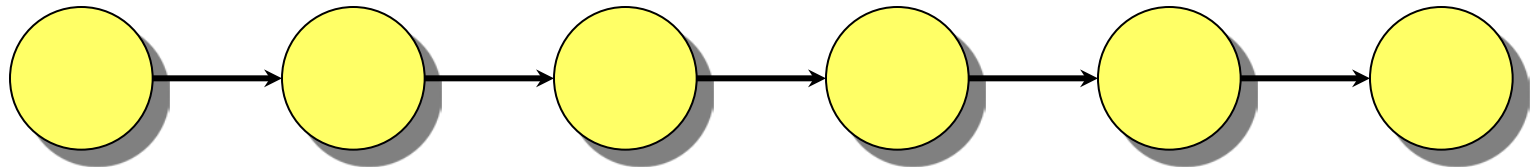
Theorem. A subpath of a shortest path is a shortest path.



Optimal substructure

Theorem. A subpath of a shortest path is a shortest path.

Proof. Cut and paste:

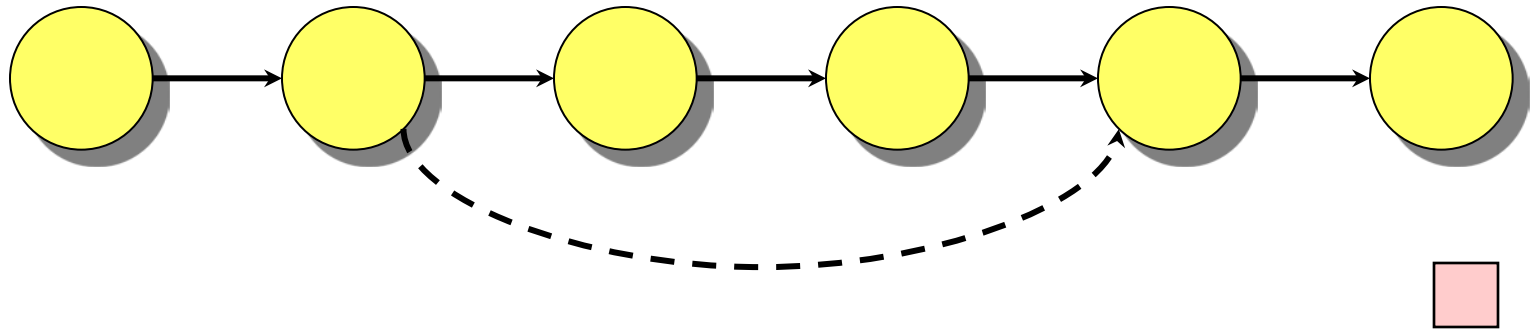


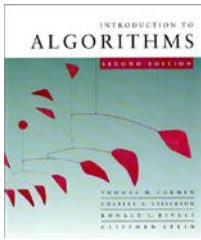


Optimal substructure

Theorem. A subpath of a shortest path is a shortest path.

Proof. Cut and paste:





Triangle inequality

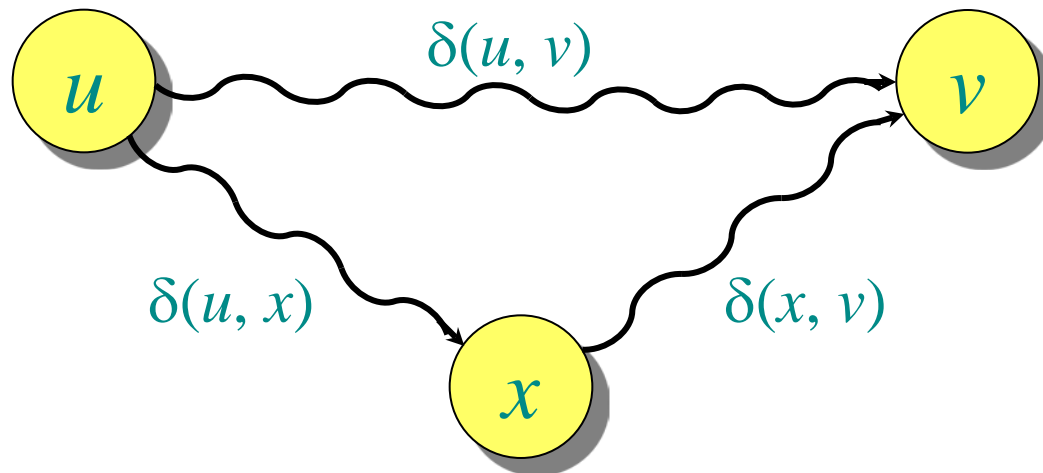
Theorem. For all $u, v, x \in V$, we have
$$\delta(u, v) \leq \delta(u, x) + \delta(x, v).$$



Triangle inequality

Theorem. For all $u, v, x \in V$, we have
$$\delta(u, v) \leq \delta(u, x) + \delta(x, v).$$

Proof.





Single-source shortest paths (nonnegative edge weights)

Problem. Assume that $w(u, v) \geq 0$ for all $(u, v) \in E$. (Hence, all shortest-path weights must exist.) From a given source vertex $s \in V$, find the shortest-path weights $\delta(s, v)$ for all $v \in V$.

IDEA: Greedy.

1. Maintain a set S of vertices whose shortest-path distances from s are known.
2. At each step, add to S the vertex $v \in V - S$ whose distance estimate from s is minimum.
3. Update the distance estimates of vertices adjacent to v .



Dijkstra's algorithm

$d[s] \leftarrow 0$

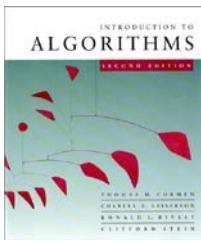
for each $v \in V - \{s\}$

do $d[v] \leftarrow \infty$

$S \leftarrow \emptyset$

$Q \leftarrow V$

▷ Q is a priority queue maintaining $V - S$,
keyed on $d[v]$



Dijkstra's algorithm

$d[s] \leftarrow 0$

for each $v \in V - \{s\}$

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$S \leftarrow \emptyset$

$Q \leftarrow V$

▷ Q is a priority queue maintaining $V - S$,
keyed on $d[v]$

while $Q \neq \emptyset$

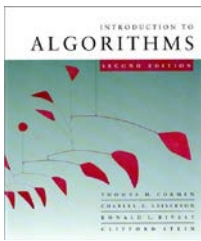
do $u \leftarrow \text{EXTRACT-MIN}(Q)$

$S \leftarrow S \cup \{u\}$

for each $v \in \text{Adj}[u]$

do if $d[v] > d[u] + w(u, v)$

then $d[v] \leftarrow d[u] + w(u, v)$



Dijkstra's algorithm

$d[s] \leftarrow 0$

for each $v \in V - \{s\}$

do $d[v] \leftarrow \infty$

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then $d[v] \leftarrow d[u] + w(u, v)$

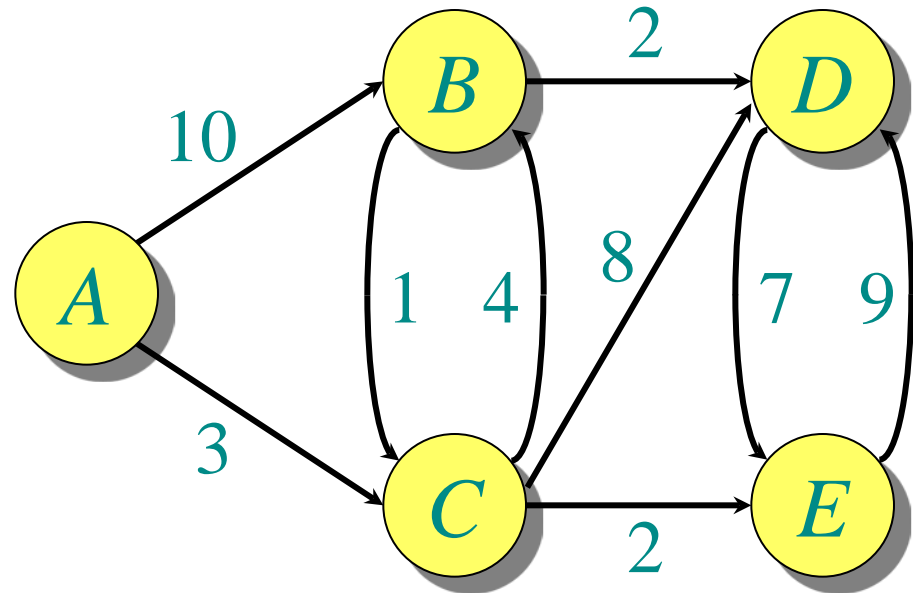
*relaxation
step*

↑
Implicit DECREASE-KEY



Example of Dijkstra's algorithm

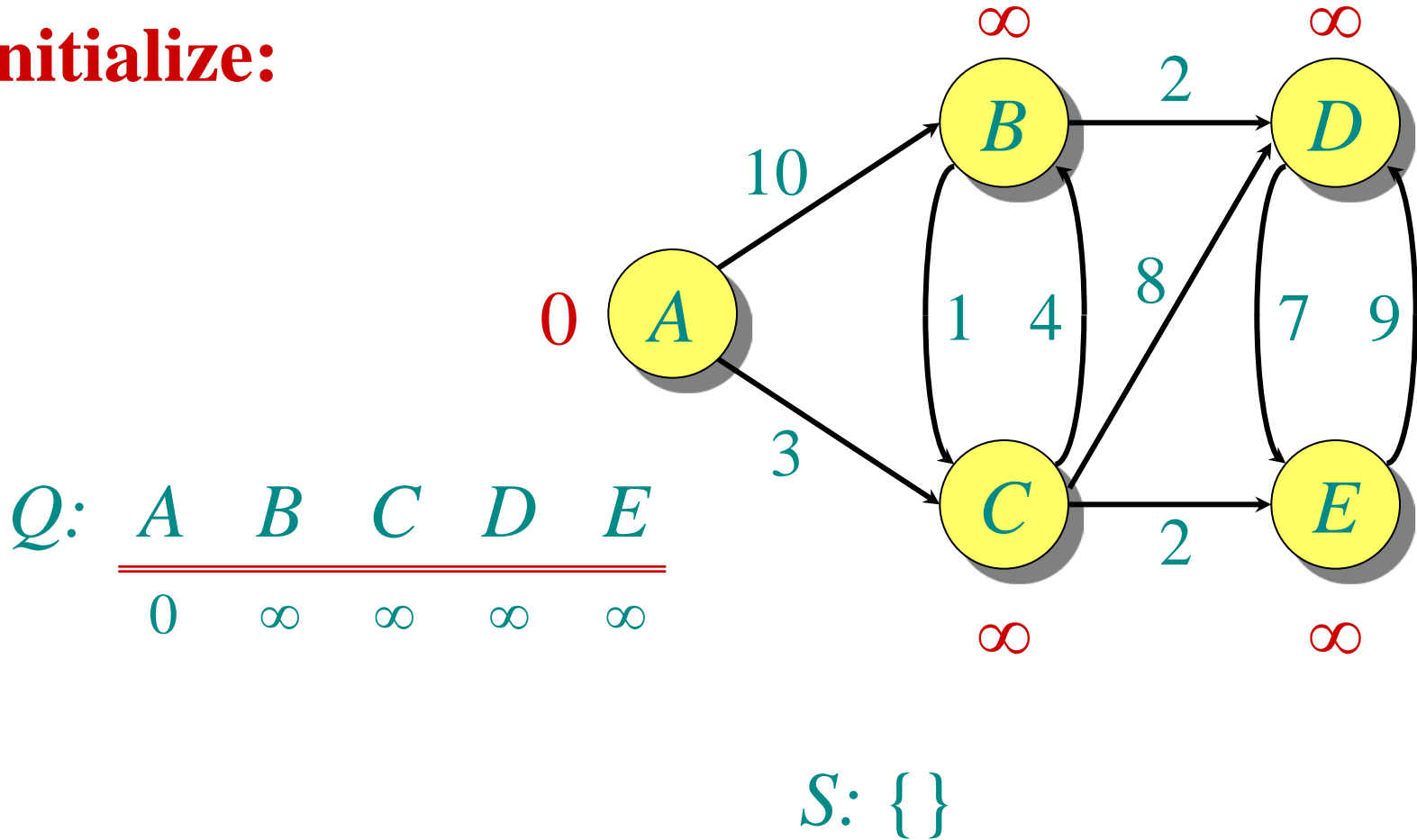
Graph with nonnegative edge weights:





Example of Dijkstra's algorithm

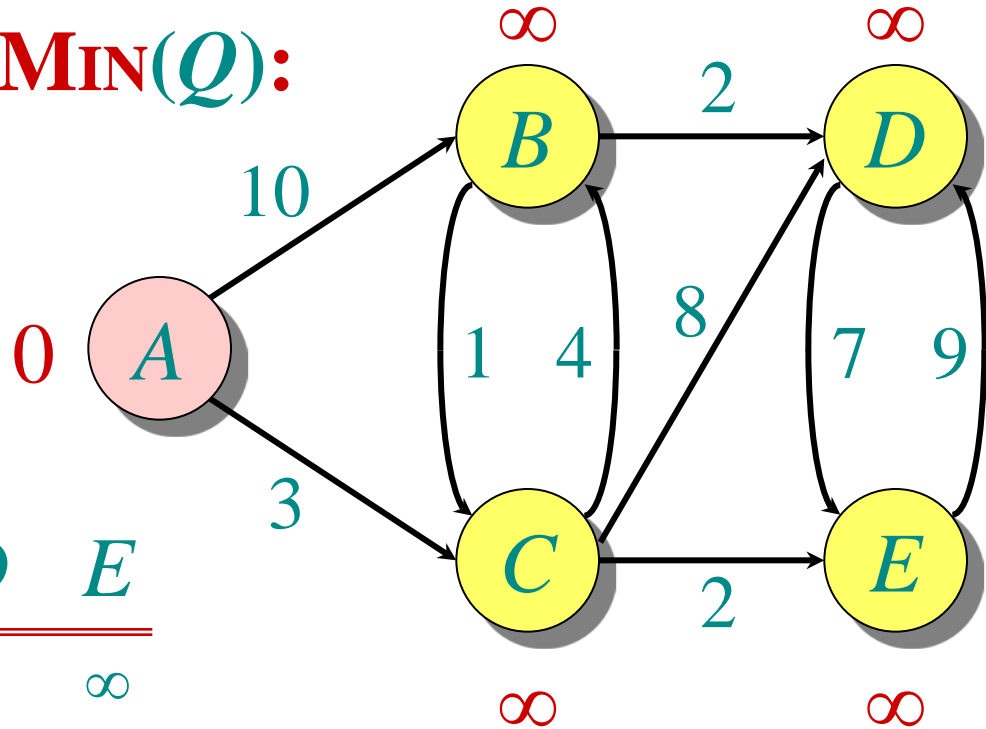
Initialize:





Example of Dijkstra's algorithm

“A” ← **EXTRACT-MIN**(Q):



Q:

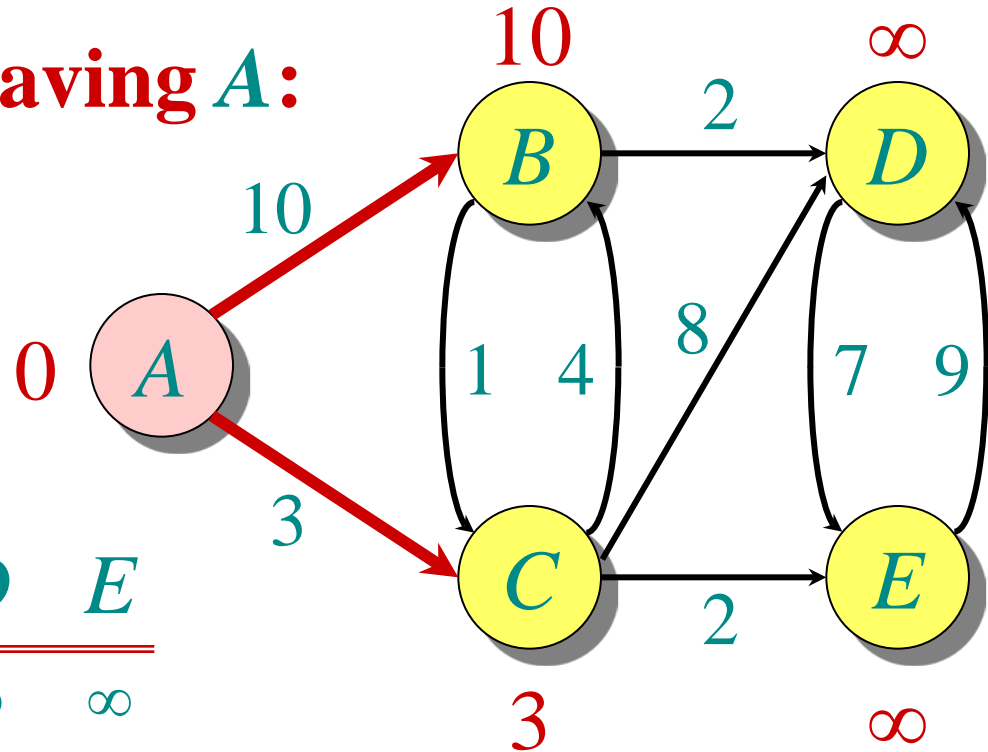
A	B	C	D	E
0	∞	∞	∞	∞

S: { A }



Example of Dijkstra's algorithm

Relax all edges leaving A :



Q :

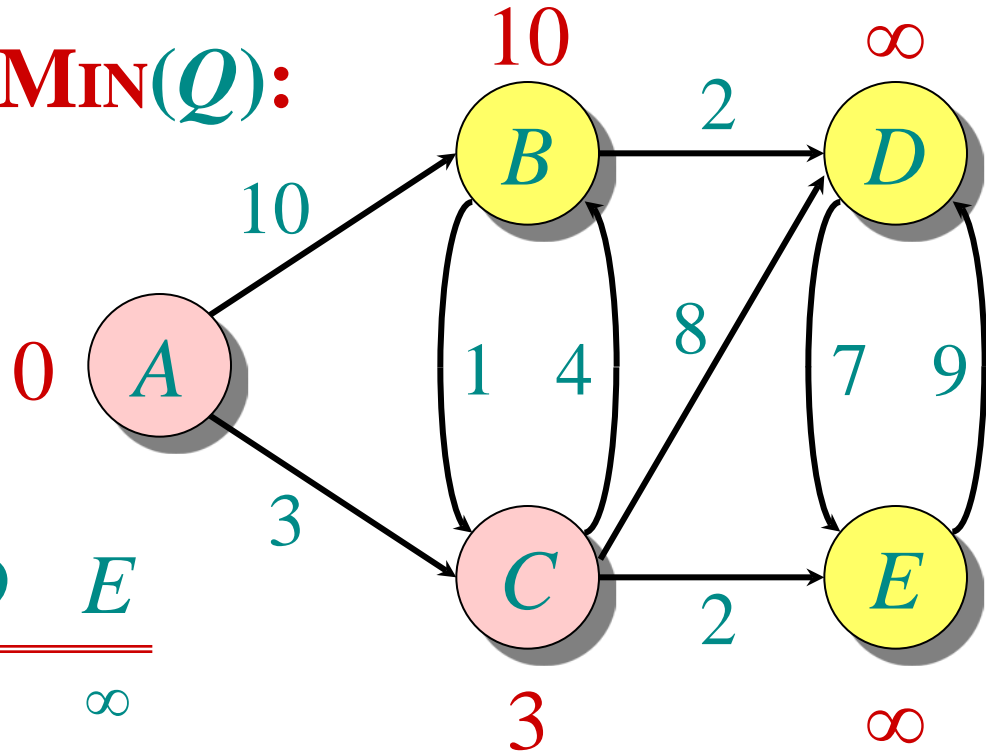
A	B	C	D	E
0	∞	∞	∞	∞
	10	3	∞	∞

$S: \{ A \}$



Example of Dijkstra's algorithm

“C” ← **EXTRACT-MIN(Q)**:



Q:

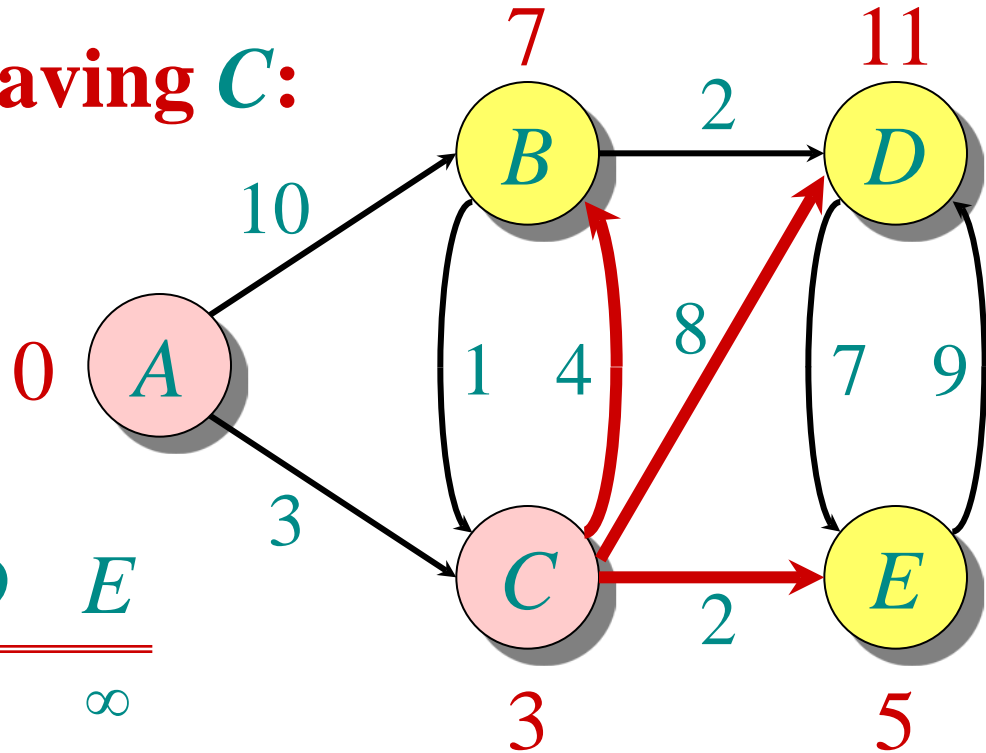
A	B	C	D	E
0	∞	∞	∞	∞
	10	3	∞	∞

S: { A, C }



Example of Dijkstra's algorithm

Relax all edges leaving C :



Q :

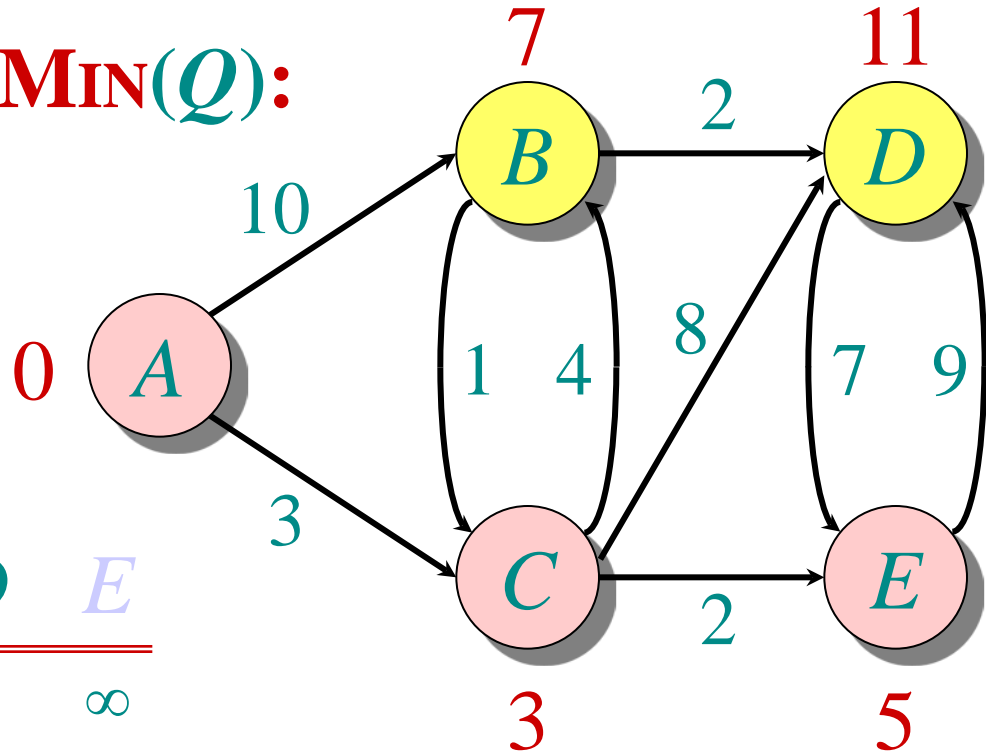
A	B	C	D	E
0	∞	∞	∞	∞
	10	3	∞	∞
	7		11	5

$S: \{ A, C \}$



Example of Dijkstra's algorithm

“E” ← **EXTRACT-MIN(Q)**:



Q:

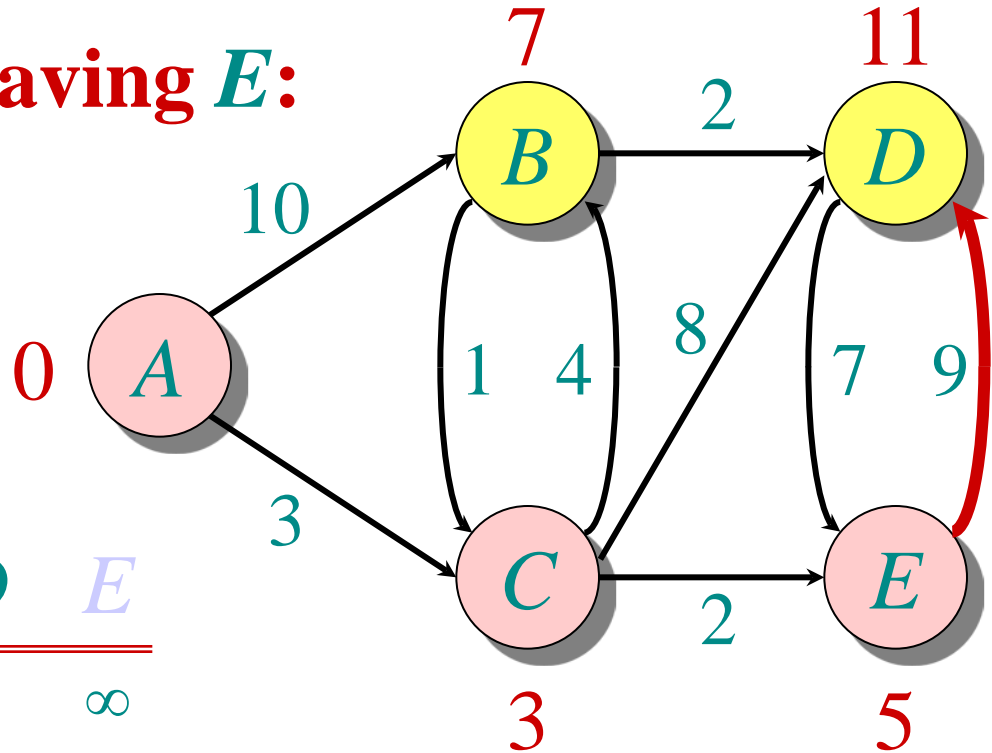
A	B	C	D	E
0	∞	∞	∞	∞
	10	3	∞	∞
	7		11	5

S: { A, C, E }



Example of Dijkstra's algorithm

Relax all edges leaving E :



Q :

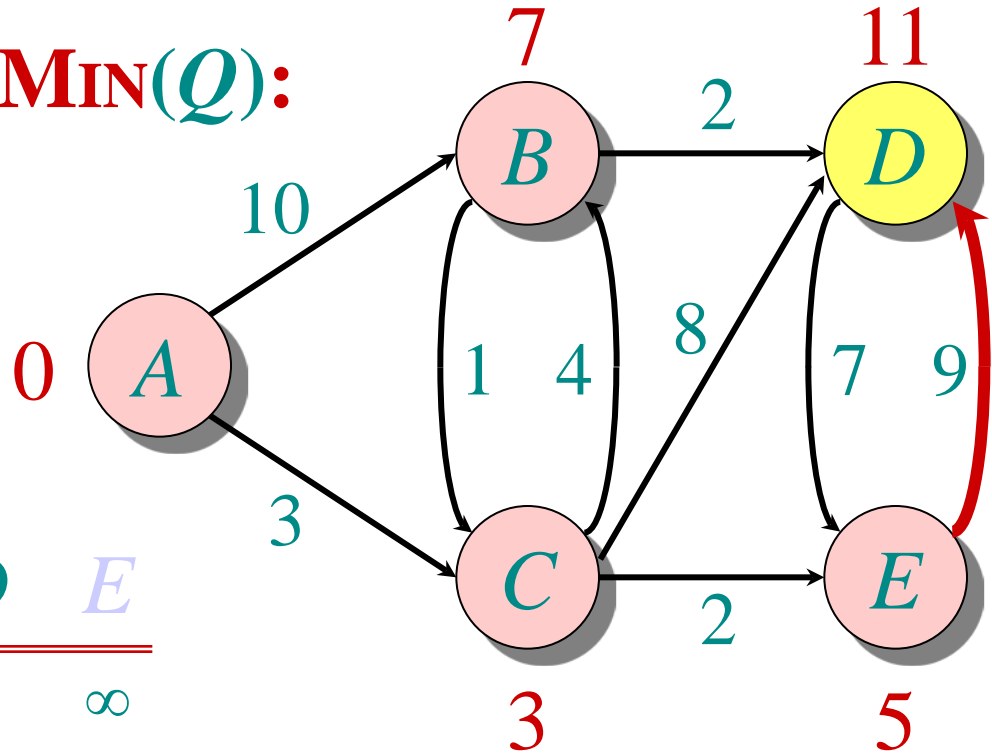
A	B	C	D	E
0	∞	∞	∞	∞
	10	3	∞	∞
	7		11	5
	7		11	

$S: \{ A, C, E \}$



Example of Dijkstra's algorithm

“B” ← **EXTRACT-MIN(Q)**:



Q:

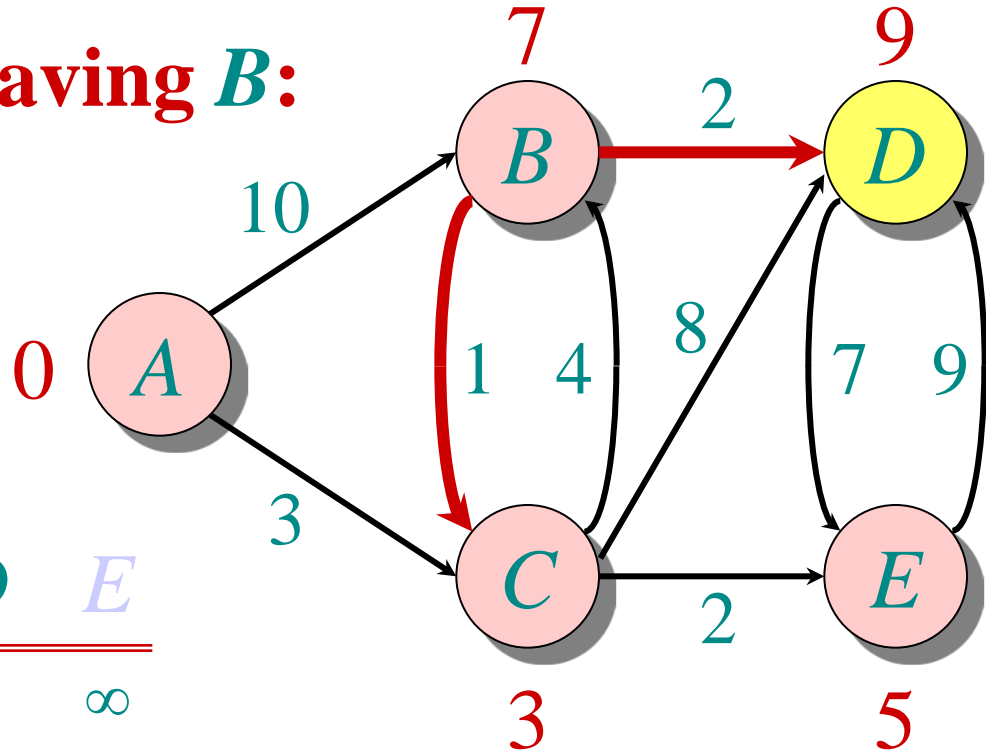
A	B	C	D	E
0	∞	∞	∞	∞
	10	3	∞	∞
	7		11	5
	7		11	

S: { A, C, E, B }



Example of Dijkstra's algorithm

Relax all edges leaving B :



Q :

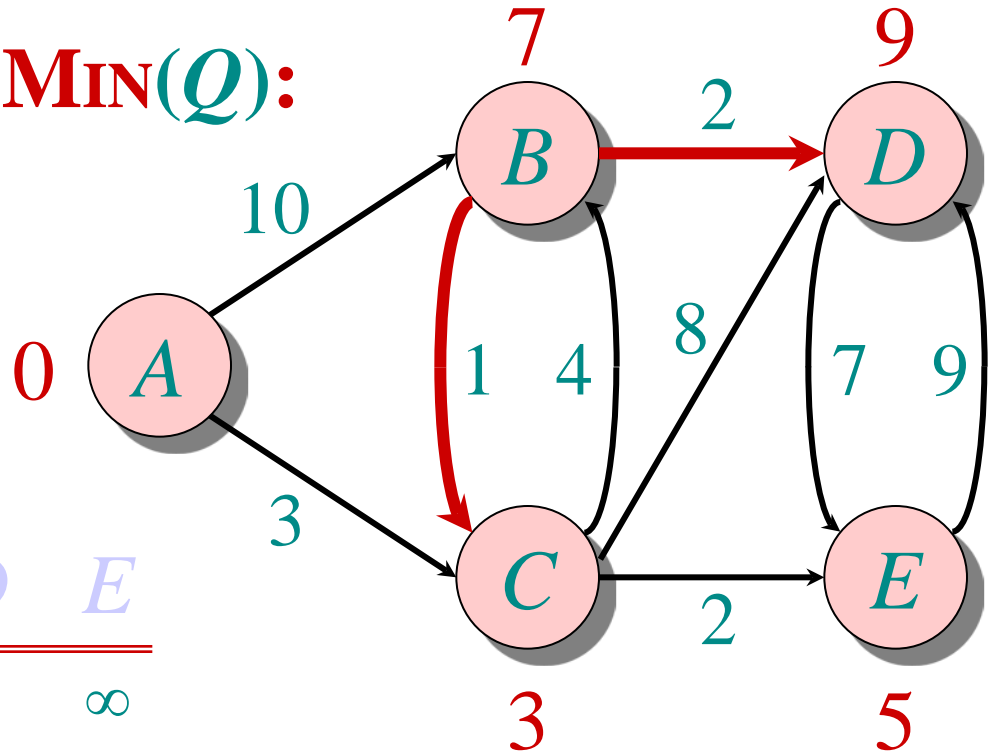
A	B	C	D	E
0	∞	∞	∞	∞
	10	3	∞	∞
	7		11	5
	7		11	
			9	

$S: \{ A, C, E, B \}$



Example of Dijkstra's algorithm

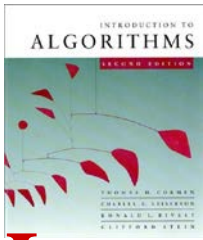
“D” ← **EXTRACT-MIN(Q)**:



Q:

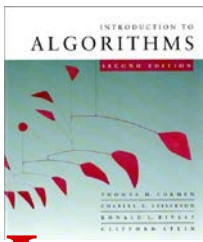
A	B	C	D	E
0	∞	∞	∞	∞
	10	3	∞	∞
	7		11	5
	7		11	
			9	

S: { A, C, E, B, D }



Correctness — Part I

Lemma. Initializing $d[s] \leftarrow 0$ and $d[v] \leftarrow \infty$ for all $v \in V - \{s\}$ establishes $d[v] \geq \delta(s, v)$ for all $v \in V$, and this invariant is maintained over any sequence of relaxation steps.



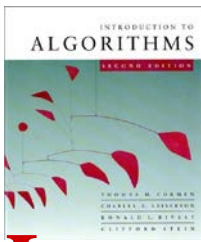
Correctness — Part I

Lemma. Initializing $d[s] \leftarrow 0$ and $d[v] \leftarrow \infty$ for all $v \in V - \{s\}$ establishes $d[v] \geq \delta(s, v)$ for all $v \in V$, and this invariant is maintained over any sequence of relaxation steps.

Proof. Suppose not. Let v be the first vertex for which $d[v] < \delta(s, v)$, and let u be the vertex that caused $d[v]$ to change: $d[v] = d[u] + w(u, v)$. Then,

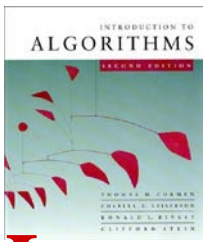
$d[v] < \delta(s, v)$	supposition
$\leq \delta(s, u) + \delta(u, v)$	triangle inequality
$\leq \delta(s, u) + w(u, v)$	sh. path \leq specific path
$\leq d[u] + w(u, v)$	v is first violation

Contradiction. □



Correctness — Part II

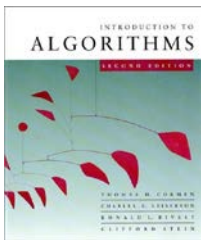
Lemma. Let u be v 's predecessor on a shortest path from s to v . Then, if $d[u] = \delta(s, u)$ and edge (u, v) is relaxed, we have $d[v] = \delta(s, v)$ after the relaxation.



Correctness — Part II

Lemma. Let u be v 's predecessor on a shortest path from s to v . Then, if $d[u] = \delta(s, u)$ and edge (u, v) is relaxed, we have $d[v] = \delta(s, v)$ after the relaxation.

Proof. Observe that $\delta(s, v) = \delta(s, u) + w(u, v)$. Suppose that $d[v] > \delta(s, v)$ before the relaxation. (Otherwise, we're done.) Then, the test $d[v] > d[u] + w(u, v)$ succeeds, because $d[v] > \delta(s, v) = \delta(s, u) + w(u, v) = d[u] + w(u, v)$, and the algorithm sets $d[v] = d[u] + w(u, v) = \delta(s, v)$. ◻



Correctness — Part III

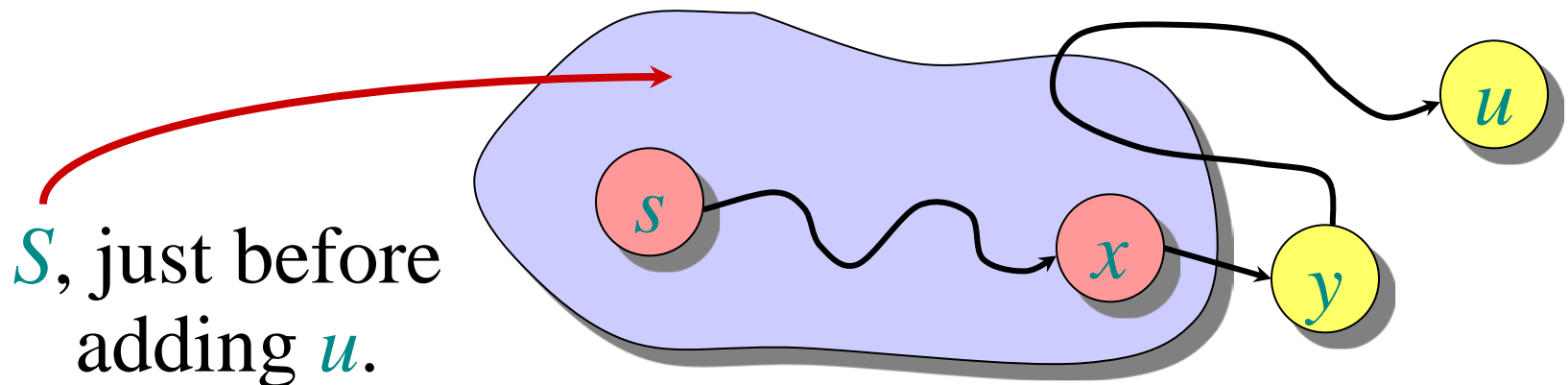
Theorem. Dijkstra's algorithm terminates with $d[v] = \delta(s, v)$ for all $v \in V$.



Correctness — Part III

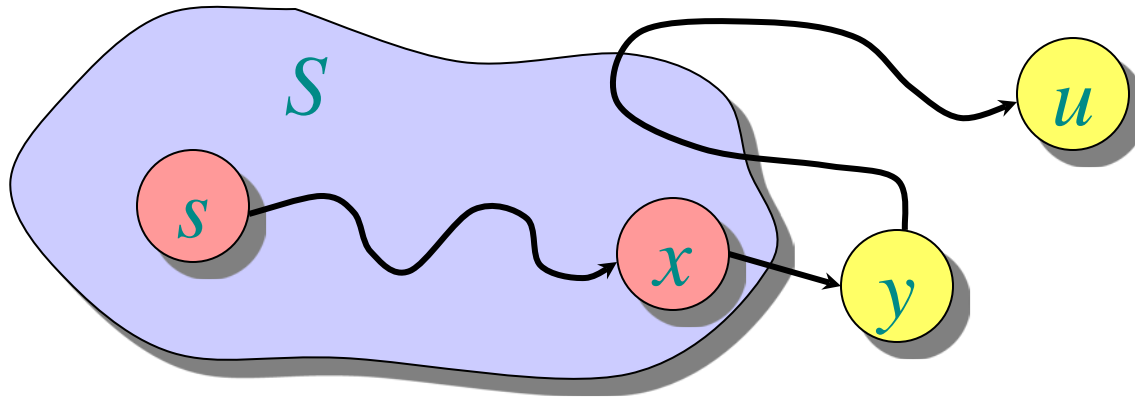
Theorem. Dijkstra's algorithm terminates with $d[v] = \delta(s, v)$ for all $v \in V$.

Proof. It suffices to show that $d[v] = \delta(s, v)$ for every $v \in V$ when v is added to S . Suppose u is the first vertex added to S for which $d[u] > \delta(s, u)$. Let y be the first vertex in $V - S$ along a shortest path from s to u , and let x be its predecessor:

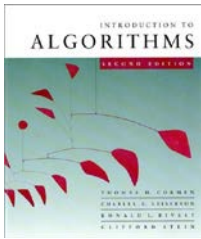




Correctness — Part III (continued)

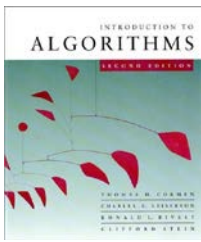


Since u is the first vertex violating the claimed invariant, we have $d[x] = \delta(s, x)$. When x was added to S , the edge (x, y) was relaxed, which implies that $d[y] = \delta(s, y) \leq \delta(s, u) < d[u]$. But, $d[u] \leq d[y]$ by our choice of u . Contradiction. □



Analysis of Dijkstra

```
while  $Q \neq \emptyset$ 
do  $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$ 
    do if  $d[v] > d[u] + w(u, v)$ 
      then  $d[v] \leftarrow d[u] + w(u, v)$ 
```



Analysis of Dijkstra

$|V|$
times

```
while  $Q \neq \emptyset$ 
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          then  $d[v] \leftarrow d[u] + w(u, v)$ 
```



Analysis of Dijkstra

$|V|$
times

$degree(u)$
times

```
while  $Q \neq \emptyset$ 
  do  $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
      $S \leftarrow S \cup \{u\}$ 
     for each  $v \in \text{Adj}[u]$ 
       do if  $d[v] > d[u] + w(u, v)$ 
          then  $d[v] \leftarrow d[u] + w(u, v)$ 
```



Analysis of Dijkstra

$|V|$ times {
while $Q \neq \emptyset$
do $u \leftarrow \text{EXTRACT-MIN}(Q)$
 $S \leftarrow S \cup \{u\}$
for each $v \in \text{Adj}[u]$
do if $d[v] > d[u] + w(u, v)$
then $d[v] \leftarrow d[u] + w(u, v)$

$\text{degree}(u)$ times {

Handshaking Lemma $\Rightarrow \Theta(E)$ implicit DECREASE-KEY's.



Analysis of Dijkstra

$|V|$ times { while $Q \neq \emptyset$
do $u \leftarrow \text{EXTRACT-MIN}(Q)$
 $S \leftarrow S \cup \{u\}$
for each $v \in \text{Adj}[u]$
do if $d[v] > d[u] + w(u, v)$
then $d[v] \leftarrow d[u] + w(u, v)$

$\text{degree}(u)$ times {

Handshaking Lemma $\Rightarrow \Theta(E)$ implicit DECREASE-KEY's.

$$\text{Time} = \Theta(V \cdot T_{\text{EXTRACT-MIN}} + E \cdot T_{\text{DECREASE-KEY}})$$

Note: Same formula as in the analysis of Prim's minimum spanning tree algorithm.



Analysis of Dijkstra (continued)

$$\text{Time} = \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
-----	--------------------------	---------------------------	-------



Analysis of Dijkstra (continued)

$$\text{Time} = \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
-----	--------------------------	---------------------------	-------

array	$O(V)$	$O(1)$	$O(V^2)$
-------	--------	--------	----------



Analysis of Dijkstra (continued)

$$\text{Time} = \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
-----	--------------------------	---------------------------	-------

array	$O(V)$	$O(1)$	$O(V^2)$
-------	--------	--------	----------

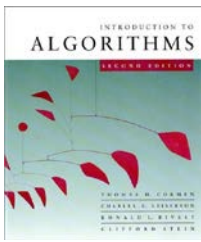
binary heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$
----------------	------------	------------	--------------



Analysis of Dijkstra (continued)

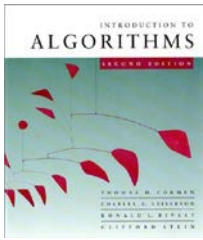
$$\text{Time} = \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$
binary heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$
Fibonacci heap	$O(\lg V)$ amortized	$O(1)$ amortized	$O(E + V \lg V)$ worst case



Unweighted graphs

Suppose that $w(u, v) = 1$ for all $(u, v) \in E$.
Can Dijkstra's algorithm be improved?

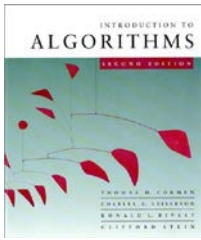


Unweighted graphs

Suppose that $w(u, v) = 1$ for all $(u, v) \in E$.

Can Dijkstra's algorithm be improved?

- Use a simple FIFO queue instead of a priority queue.



Unweighted graphs

Suppose that $w(u, v) = 1$ for all $(u, v) \in E$.
Can Dijkstra's algorithm be improved?

- Use a simple FIFO queue instead of a priority queue.

Breadth-first search

```
while  $Q \neq \emptyset$ 
  do  $u \leftarrow \text{DEQUEUE}(Q)$ 
    for each  $v \in \text{Adj}[u]$ 
      do if  $d[v] = \infty$ 
          then  $d[v] \leftarrow d[u] + 1$ 
              ENQUEUE( $Q, v$ )
```



Unweighted graphs

Suppose that $w(u, v) = 1$ for all $(u, v) \in E$.
Can Dijkstra's algorithm be improved?

- Use a simple FIFO queue instead of a priority queue.

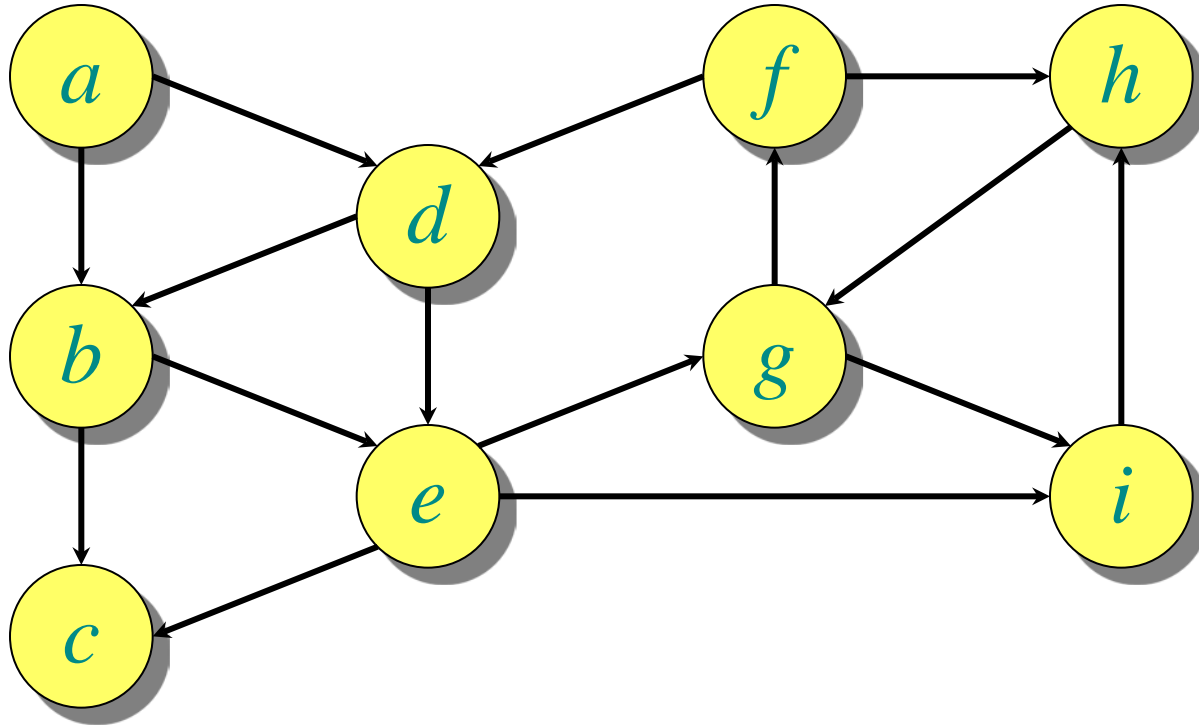
Breadth-first search

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       do if  $d[v] = \infty$ 
          then  $d[v] \leftarrow d[u] + 1$ 
              ENQUEUE( $Q, v$ )
```

Analysis: Time = $O(V + E)$.



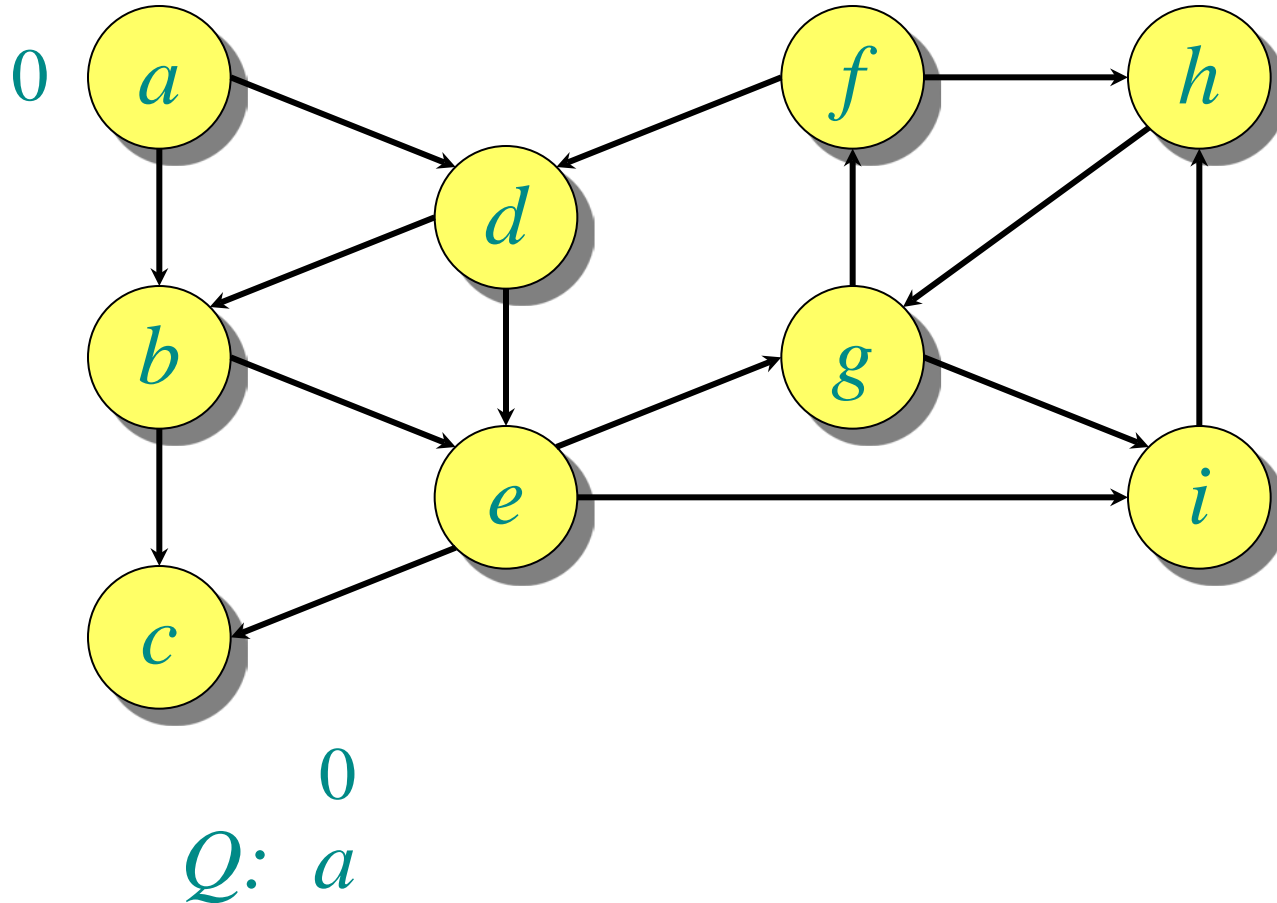
Example of breadth-first search



Q:

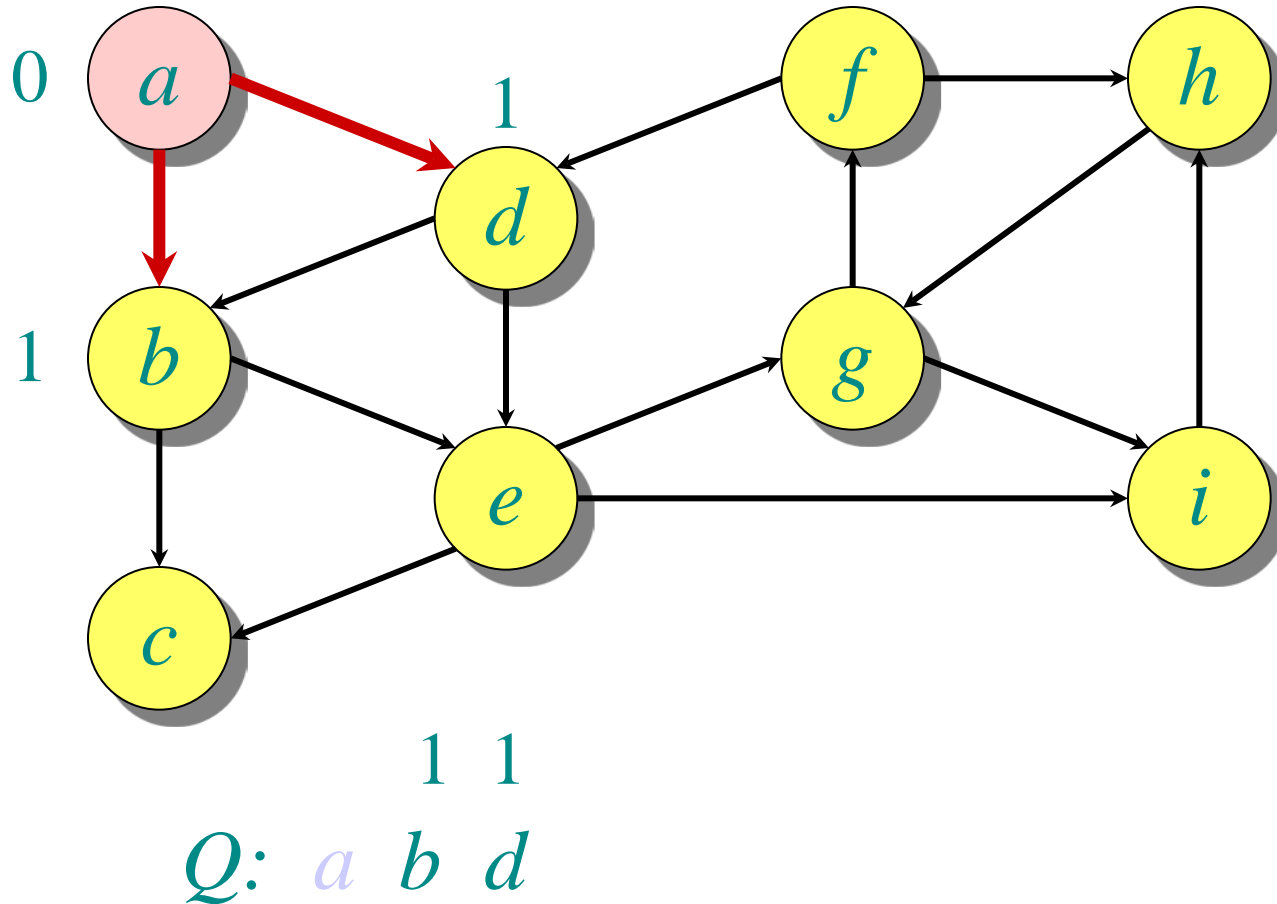


Example of breadth-first search



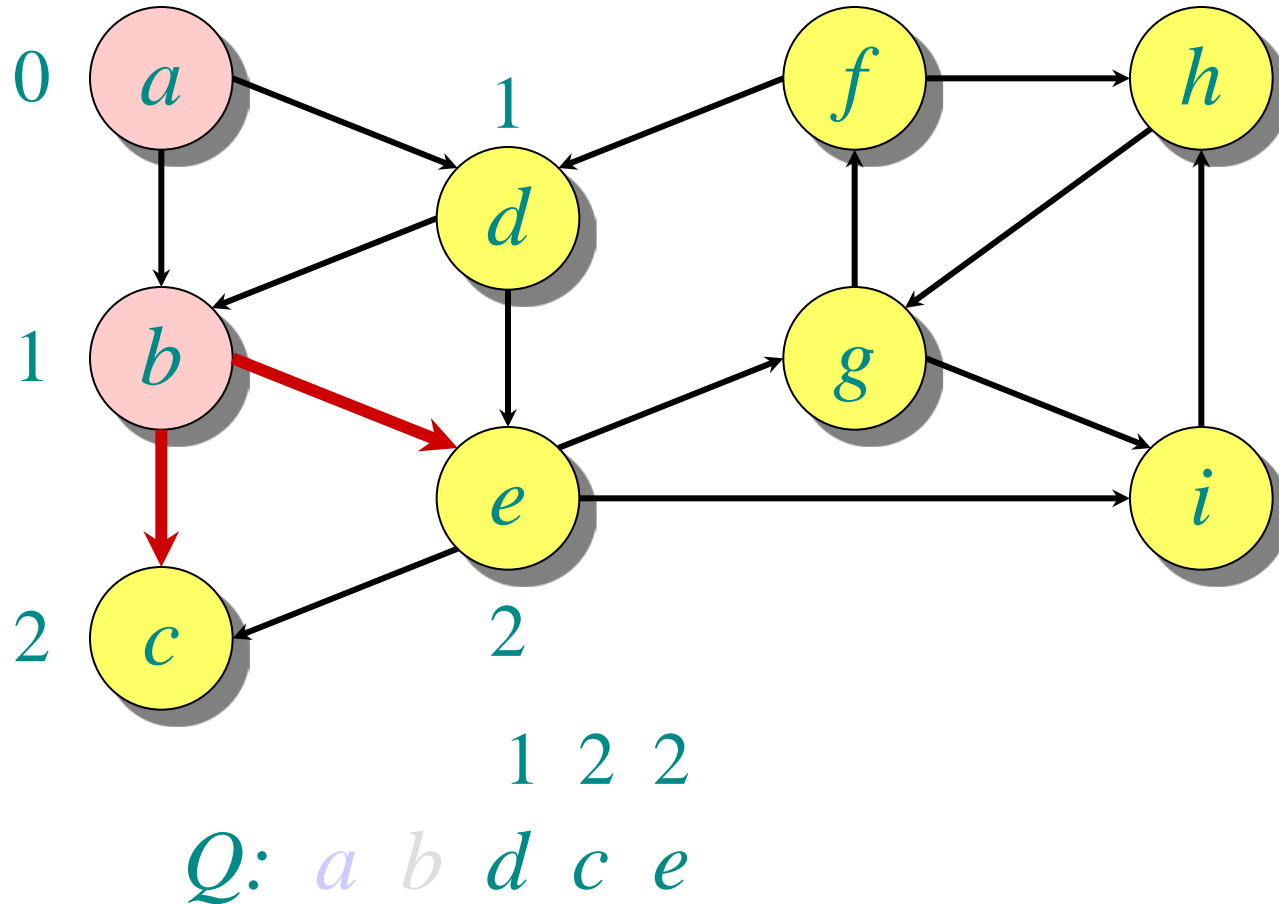


Example of breadth-first search



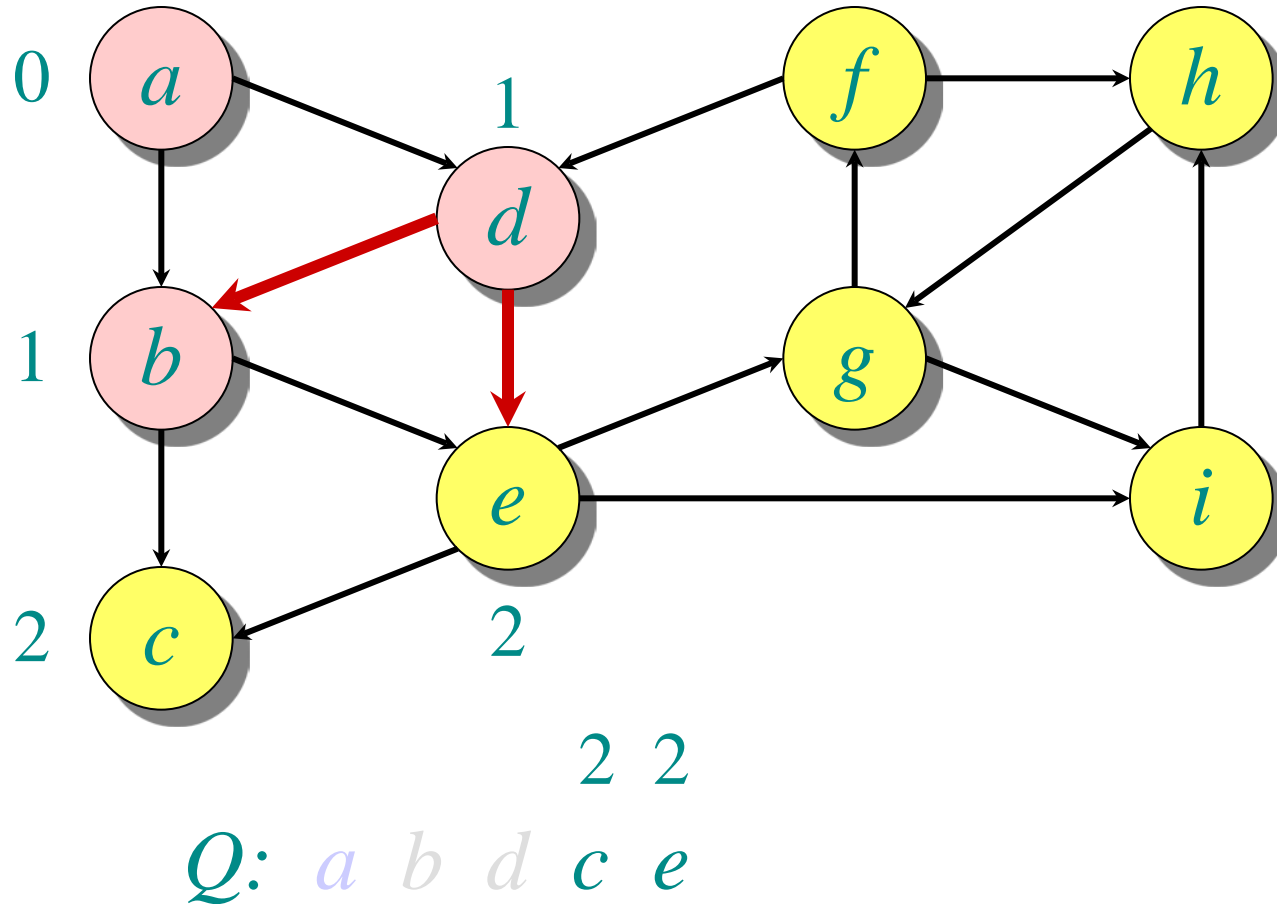


Example of breadth-first search



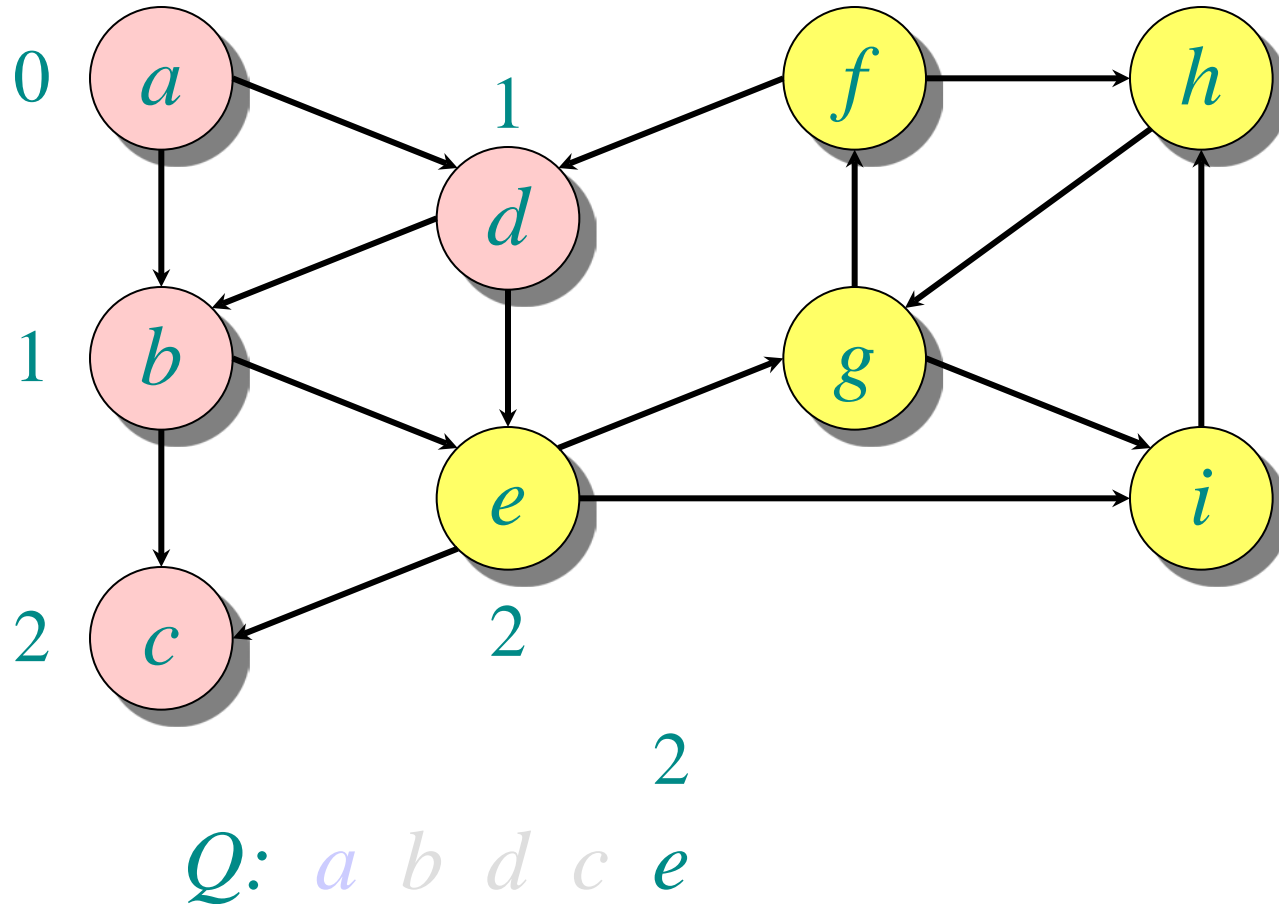


Example of breadth-first search



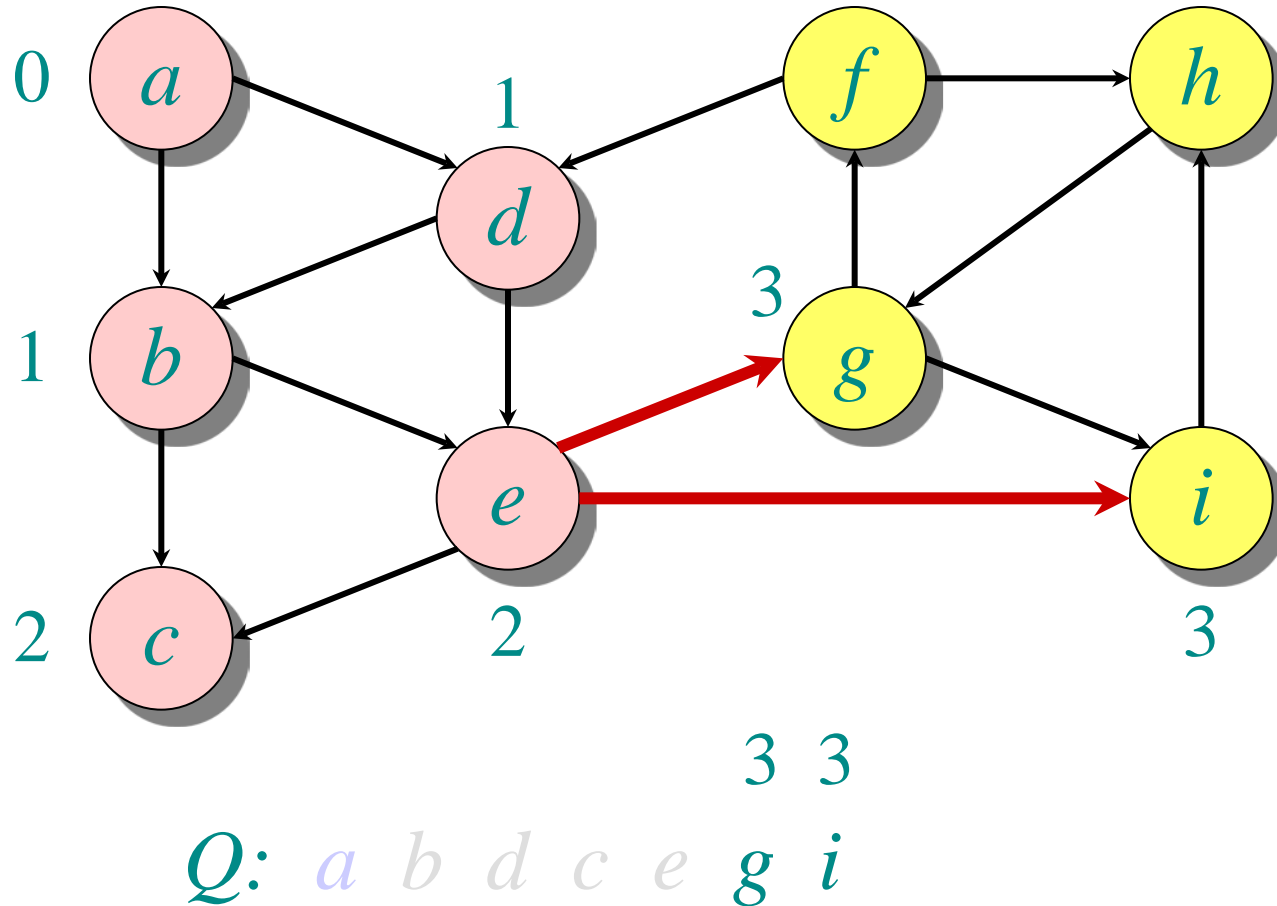


Example of breadth-first search



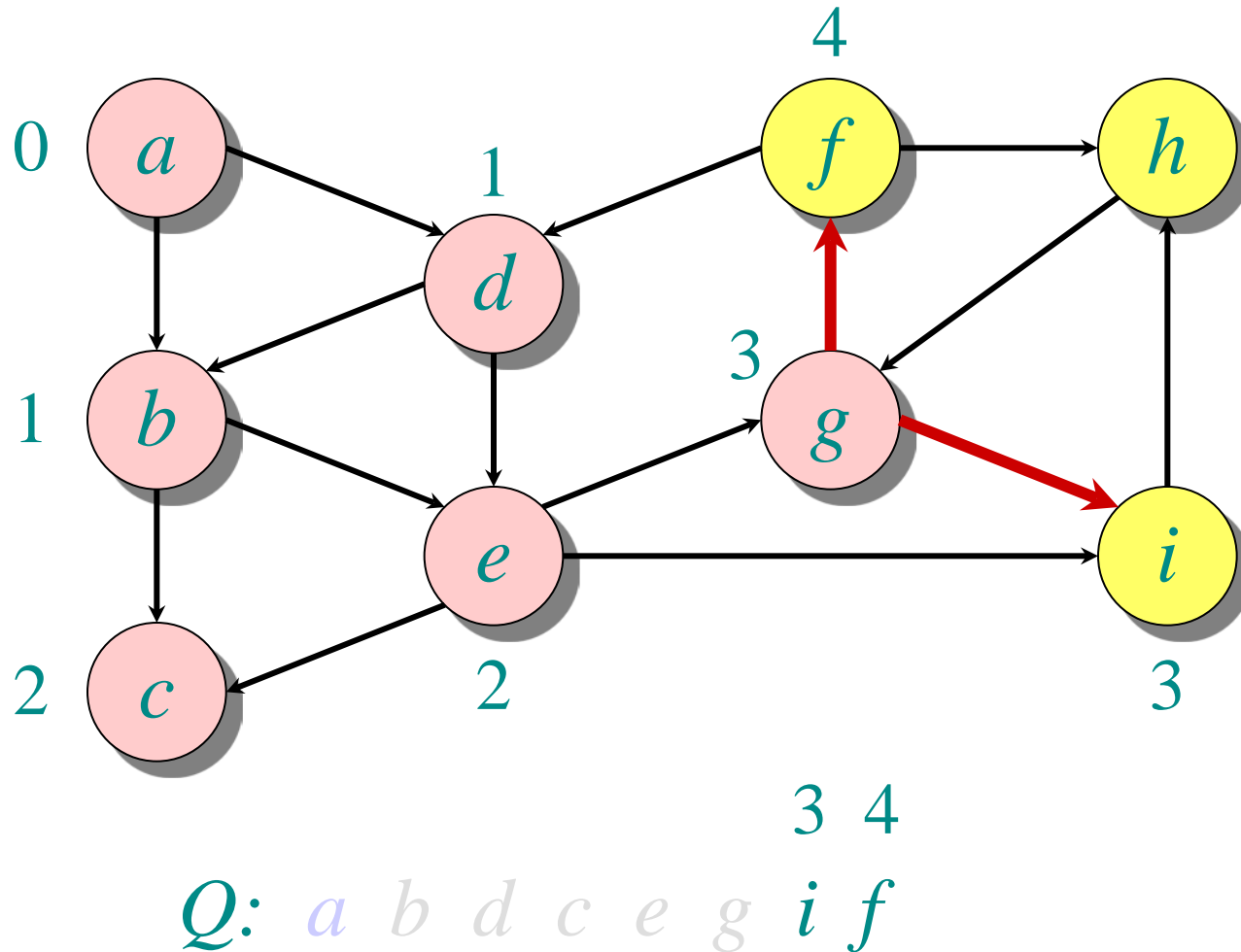


Example of breadth-first search



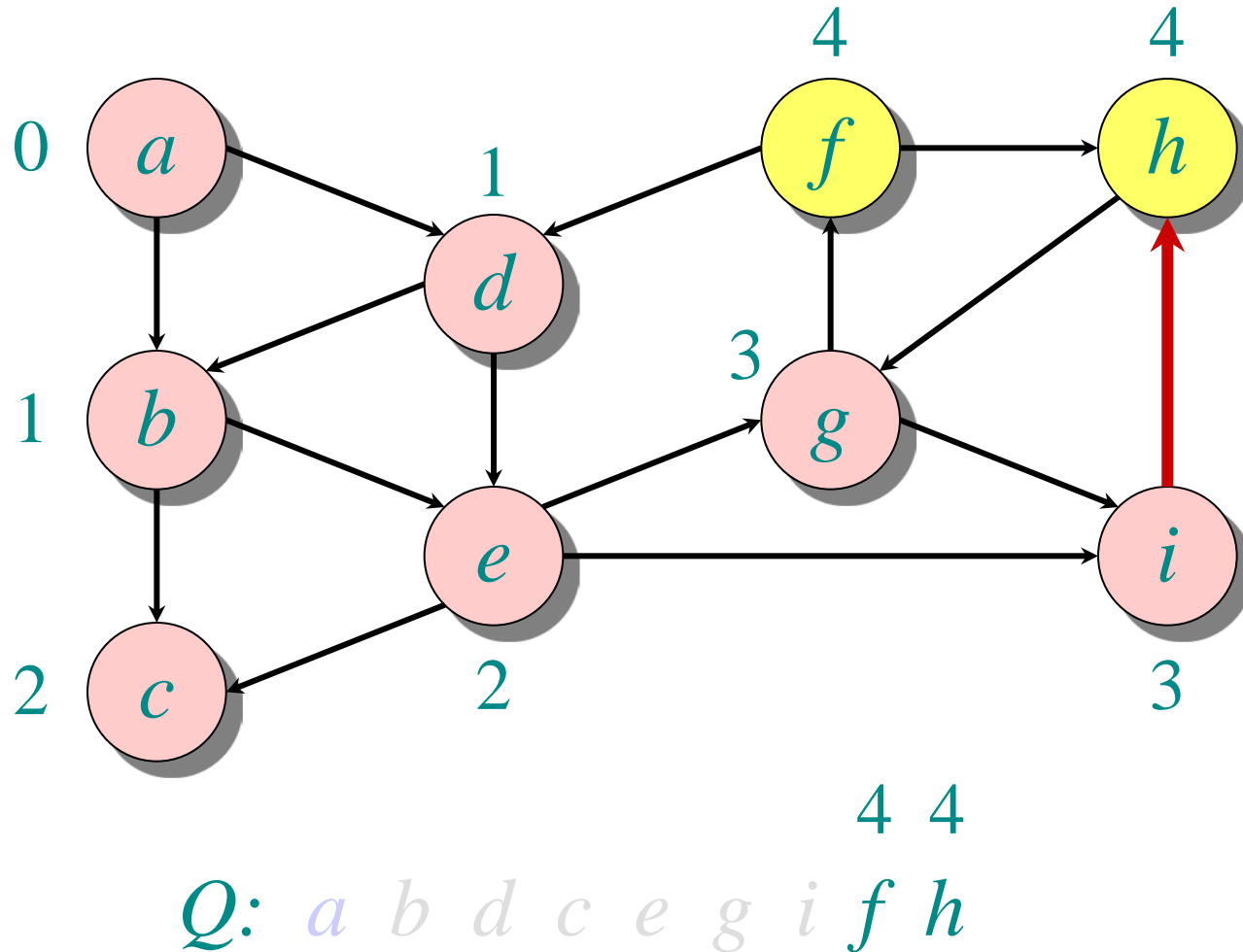


Example of breadth-first search



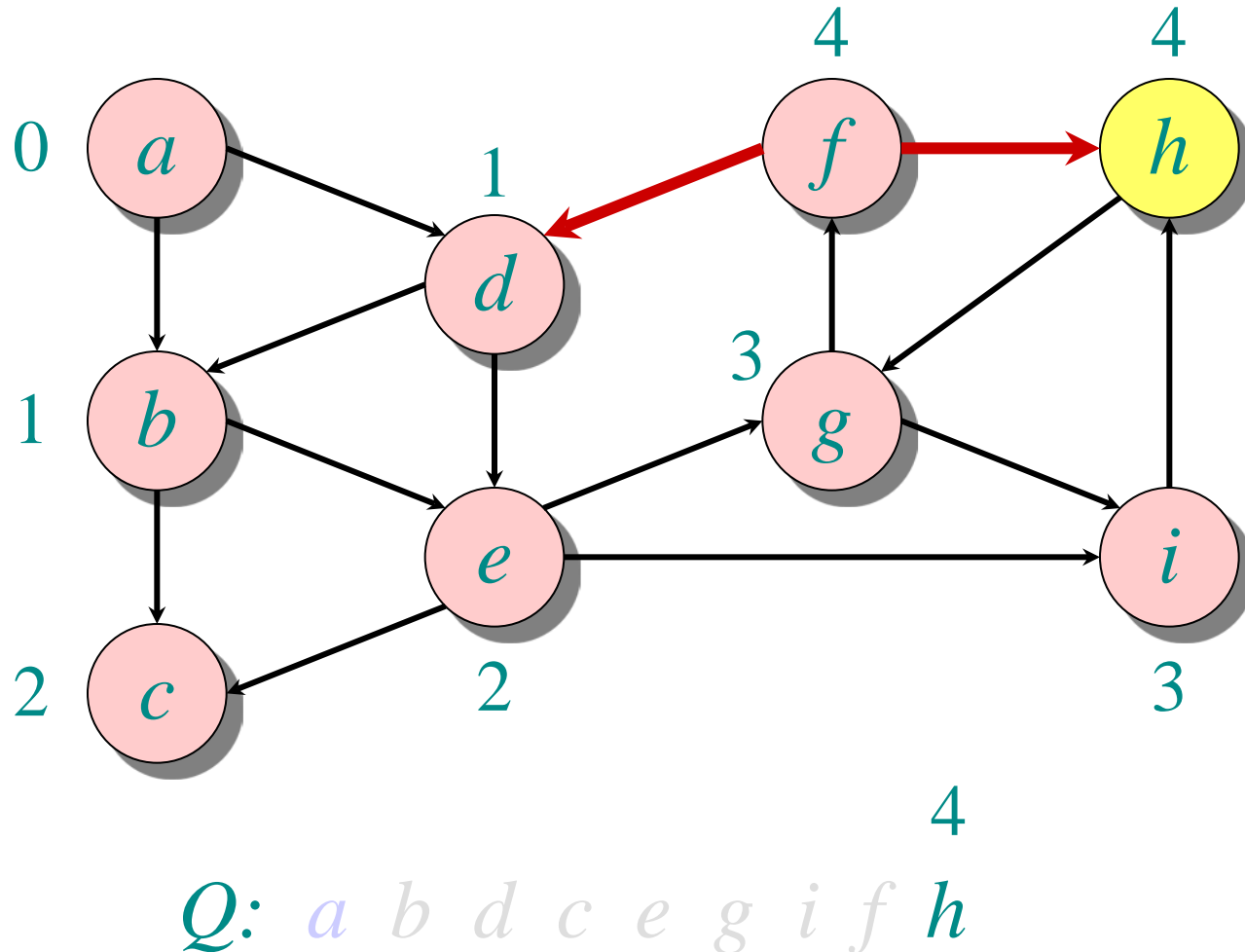


Example of breadth-first search



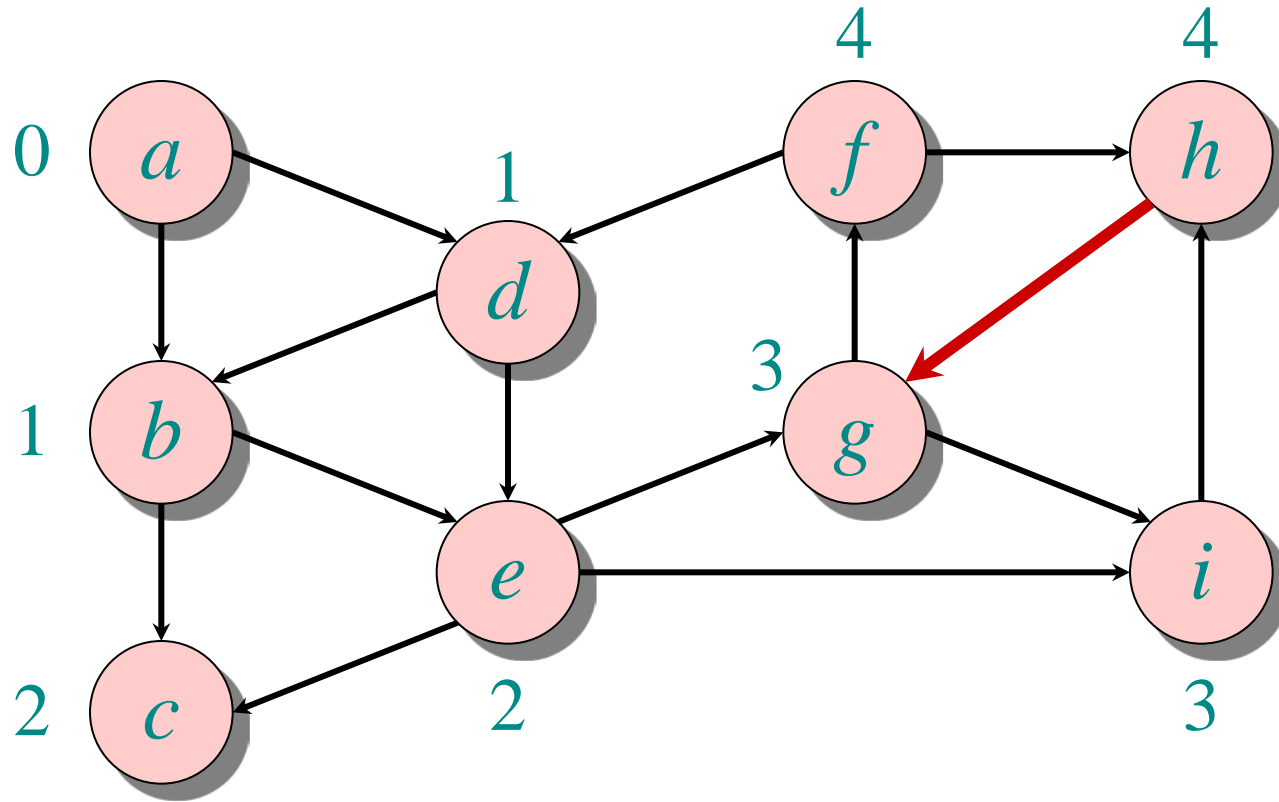


Example of breadth-first search





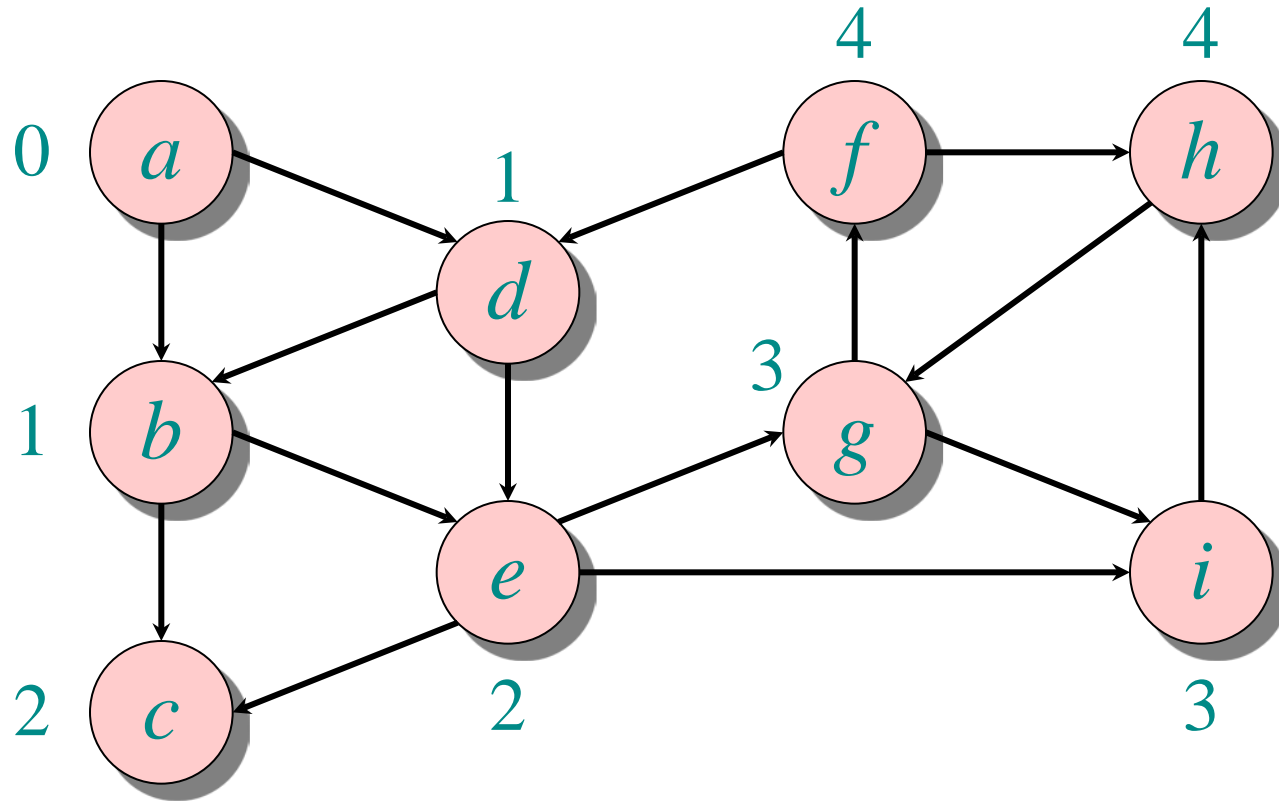
Example of breadth-first search



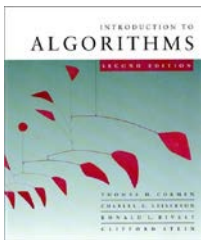
Q: *a b d c e g i f h*



Example of breadth-first search



Q: a b d c e g i f h



Correctness of BFS

```
while  $Q \neq \emptyset$ 
do  $u \leftarrow \text{DEQUEUE}(Q)$ 
  for each  $v \in \text{Adj}[u]$ 
  do if  $d[v] = \infty$ 
    then  $d[v] \leftarrow d[u] + 1$ 
      ENQUEUE( $Q, v$ )
```

Key idea:

The FIFO Q in breadth-first search mimics the priority queue Q in Dijkstra.

- **Invariant:** v comes after u in Q implies that $d[v] = d[u]$ or $d[v] = d[u] + 1$.